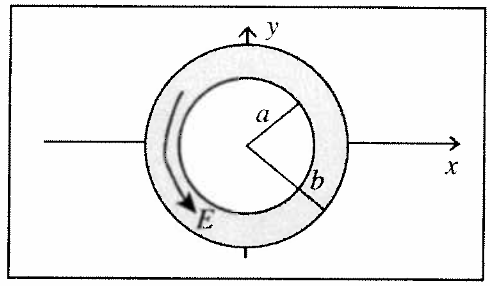
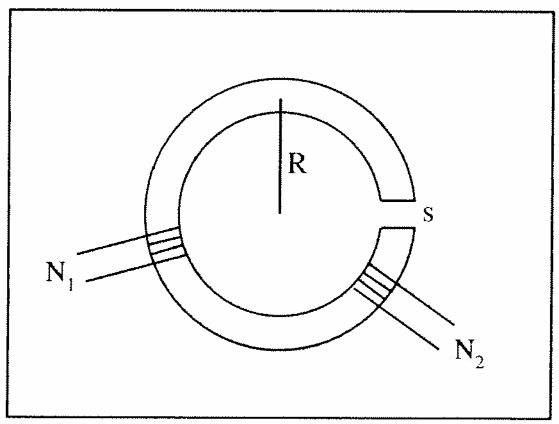
Electricity and Magnetism - Final Exam - Spring 2013

Work four of the six problems. Place the problems in the order you wish them graded. The first two problems form the first half test; the second two problems form the second half test. If you turn in all six problems, then 75% of your score on the last two problems will be used to replace your lowest test score (for better of worse).

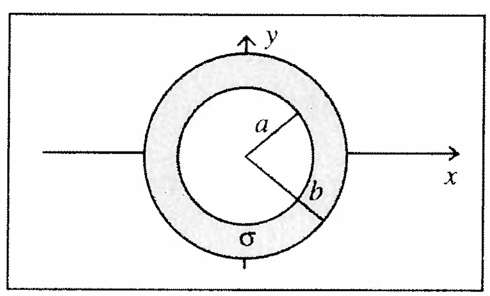
Problem 4.1 A metal washer, a thin metal disk with inner radius a, outer radius b, and thickness ℓ, is in the $x-y$ plane centered at the origin. The metal has conductivity $\sigma(s) = \gamma/s$ where γ is a constant. The region containing the disk also contains an electric field $\vec{E} = E_0\hat{\phi}$ where E_0 is a constant. Compute the magnetic field at the origin that results from the current in the disk produced by the given electric field. You may treat the current as a surface current since the disk is thin.



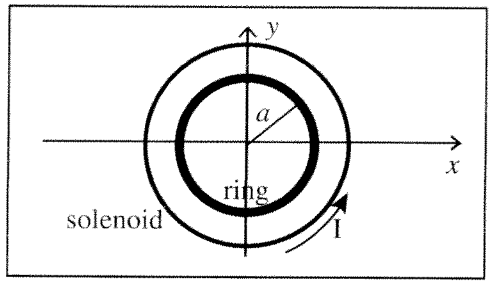
Problem 4.2 The figure below shows two coils of wire with number of turns N_1 and N_2 wound on an iron ring with relative permeability μ_r, radius R, and cross-sectional area A. The iron ring has a small gap of width s which allows the mutual inductance of the two coils to be adjusted. Compute the mutual inductance of the two coils.



Problem 4.3 A thin circular disk with inner radius a and outer radius b is in the $x-y$ plane centered at the origin. The disk has a surface charge density $\sigma(s) = \gamma s^2$ where γ is a constant. Compute the electric potential at the origin.



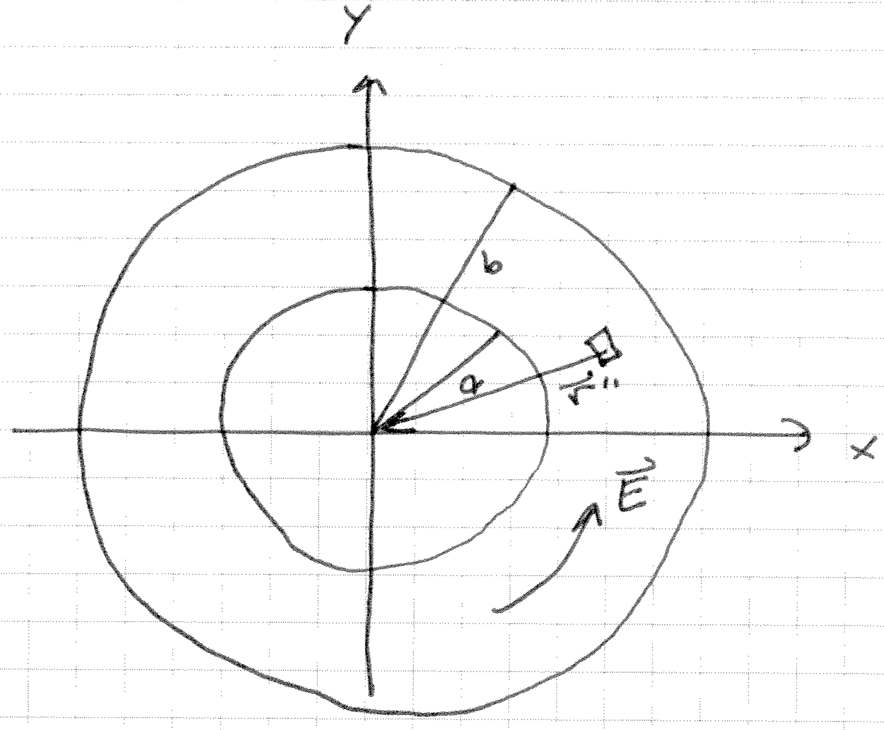
Problem 4.4 A long solenoid is wound with $N = 100$ turns over a distance $\ell = 20$cm. At time $t = 0$, the solenoid carries a current $I_0 = \frac{1}{4}A$. My left hand is in the solenoid adjusting, for some reason, a compass at the center of the solenoid. My wedding ring has radius of about $a = 1$cm and cross-sectional area of about $A = 1\text{mm}^2$. The ring is made of gold that has resistivity $\rho = 2.4 \times 10^{-8}\Omega$m. The normal of the surface bounded by the right is parallel to the axis of the solenoid. At $t = 0$, the fuse in the meter that measures the current in the solenoid blows and the current in the solenoid decreases to zero as $I(t) = I_0 e^{-t/\tau}$ where $\tau = 1 \times 10^{-3}$s. What is the peak value of the current that is induced in my ring?



Problem 4.5 A rectangular channel of width a and height b occupies the region $0 < x < a$ and $0 < y < b$. The channel is infinite in the z direction. The $y = 0$, $y = b$, and $x = a$ sides of the channel are grounded. The $x = 0$ side has potential $V(0, y) = 0$ for $y < b/4$ and for $y > 3b/4$. In between, $V(0, y) = V_0$ for $b/4 < y < 3b/4$. Compute the potential in the channel.

Problem 4.6 A long cylindrical conductor with relative permeability μ_r and radius $s = a$ carries a total current I uniformly distributed over its cross section. The wire is co-axial with the z axis and carries current in the $+\hat{z}$ direction. Outside the wire $s > a$ the current density decays exponentially and is given by $\vec{J} = \frac{J_0}{s} e^{-s/b}\hat{z}$ where J_0 and b are constants. Compute $\vec{H}$ and $\vec{B}$ everywhere, both inside and outside the wire.

4.1



The current density in the washer is

$$\vec{J} = \sigma \vec{E} = \sigma E_0 \hat{\phi}$$

The surface current density, treating the washer as thin is

$$\vec{K} = \vec{J} \cdot l = \sigma l E_0 \hat{\phi}$$

The ~~about the~~ magnetic field at the origin is given by

$$\vec{B} = \frac{\mu_0}{4\pi} \int_S \frac{\vec{K} \times \hat{r}'}{(r'')^2} \, da'$$

Field Point $\vec{r} = 0$

Source Point $\vec{r}\,' = s'\hat{s}'$

$\hat{r}' = \hat{s}'$

Displacement Vector $\vec{r}\,'' = \vec{r} - \vec{r}\,' = -s'\hat{s}'$

$r'' = s'$

$$\vec{K} \times \hat{r}' = \sigma \ell E_0 \hat{\phi} \times \hat{s}' = \sigma \ell E_0 \hat{z}$$

Biot-Savart $da' = ds'\, s'd\phi'$

$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times \hat{r}'}{(r'')^2} da'$$

$$= \frac{\mu_0 \ell E_0}{4\pi} \int_0^{2\pi} d\phi' \int_a^b ds'\, s' \frac{\left(\frac{\gamma}{s'}\right)}{(s')^2} \hat{z}$$

$$= \frac{\mu_0 \ell \gamma E_0 \hat{z} 2\pi}{4\pi} \int_a^b \frac{ds'}{s'^2}$$

$$= \frac{\mu_0 \ell \gamma E_0 \hat{z} 2\pi}{4\pi} \left(-\frac{1}{s'}\right)_a^b$$

$$= \frac{\mu_0 \ell \gamma E_0 \hat{z} 2\pi}{4\pi} \left(\frac{1}{a} - \frac{1}{b}\right) = \frac{\mu_0 \ell \gamma E_0}{2} \left(\frac{1}{a} - \frac{1}{b}\right) \hat{z}$$

(4.2) Assume a current I_1 flows in loop 1. The mmf of the ring is

$$\text{mmf} = N_1 I_1 = \int \vec{H} \cdot d\vec{l}$$

$$= H_i (2\pi R - s) + H_o \cdot s$$

H_i = inside

H_o = outside

By no magnetic monopoles, $B_i = B_o$.

$$H_o = \frac{B_o}{\mu_o} \qquad H_i = \frac{B_i}{\mu_o \mu_r} = \frac{B_o}{\mu_o \mu_r}$$

So

$$N_1 I_1 = \frac{B_o}{\mu_o \mu_r}(2\pi R - s) + \frac{B_o}{\mu_o} s$$

$$\mu_o \mu_r N_1 I_1 = B_o \left(2\pi R - s + \mu_r s \right)$$

$$B_o = \frac{\mu_o \mu_r N_1 I_1}{2\pi R + (\mu_r - 1) s}$$

The magnetic flux through loop 2 is

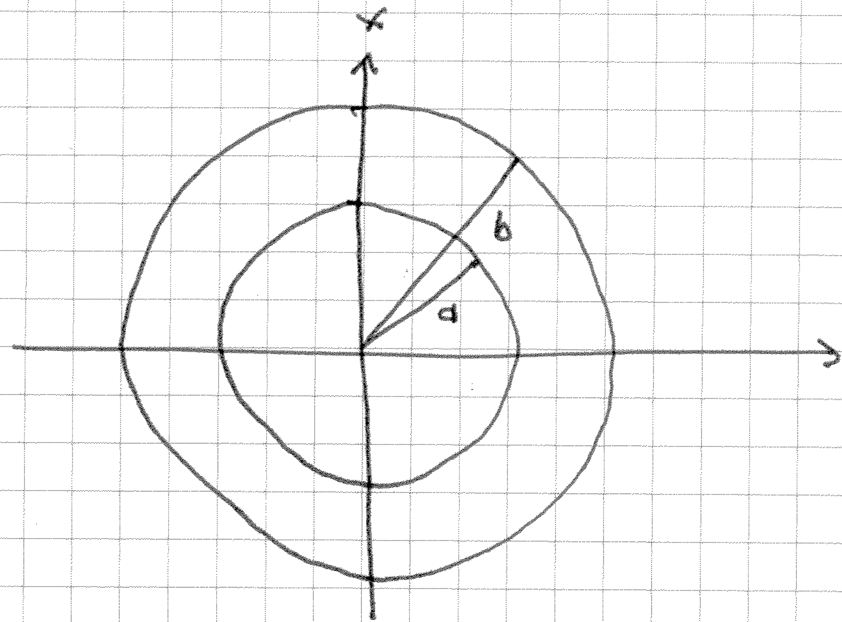
$$\Phi_2 = N_2 B A$$

$$= \frac{\mu_0 \mu_r N_1 N_2 A I_1}{2\pi R + (\mu_r - 1) s}$$

The mutual inductance is then

$$M = \frac{\Phi_2}{I_1} = \frac{\mu_0 \mu_r N_1 N_2 A}{2\pi R + (\mu_r - 1) s}$$

4.3



The electric potential is

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma \, da'}{r''} \qquad \sigma = \gamma s^2$$

$$\vec{r} = 0 \qquad \vec{r}\,' = s'\hat{s}' \qquad \vec{r}\,'' = -s'\hat{s}'$$

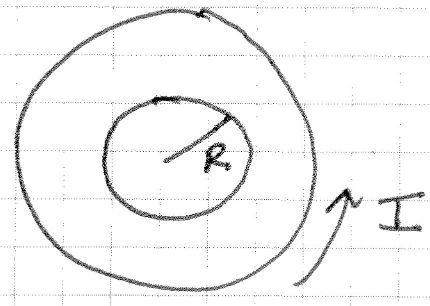
$$r'' = s'$$

$$da' = ds' \, s' d\phi'$$

$$V = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi} d\phi' \int_a^b ds' \frac{\gamma s'^2 s'}{s'}$$

$$V = \frac{2\pi\gamma}{4\pi\varepsilon_0}\left(\frac{b^3}{3} - \frac{a^3}{3}\right) = \frac{\gamma}{6\varepsilon_0}\left(b^3 - a^3\right)$$

4.4



Resistance of Wedding Ring

$$R = \frac{\rho\, l_r}{A} = \frac{\rho\, 2\pi a}{A} \qquad l_r = 2\pi a$$

Magnetic Field of Solenoid

$$B = \mu_0 \left(\frac{N}{l}\right) I$$

$$= \mu_0 \left(\frac{N}{l}\right) I_0\, e^{-t/\tau}$$

Magnetic Flux through Ring

$$\Phi_m = N_r B A_r = (1)(B)\,\pi a^2$$

$$= \mu_0 \pi a^2 \left(\frac{N}{l}\right) I_0\, e^{-t/\tau}$$

Faraday's Law - emf around loop

$$emf = -\frac{d\Phi_m}{dT} = -\frac{\mu_0 \pi a^2 N}{l\tau} I_0 e^{-t/\tau}$$

The peak emf occurs at $t=0$

$$I_{peak} = \frac{emf(0)}{R}$$

$$= \frac{\mu_0 \pi a^2 N I_0}{l \tau R}$$

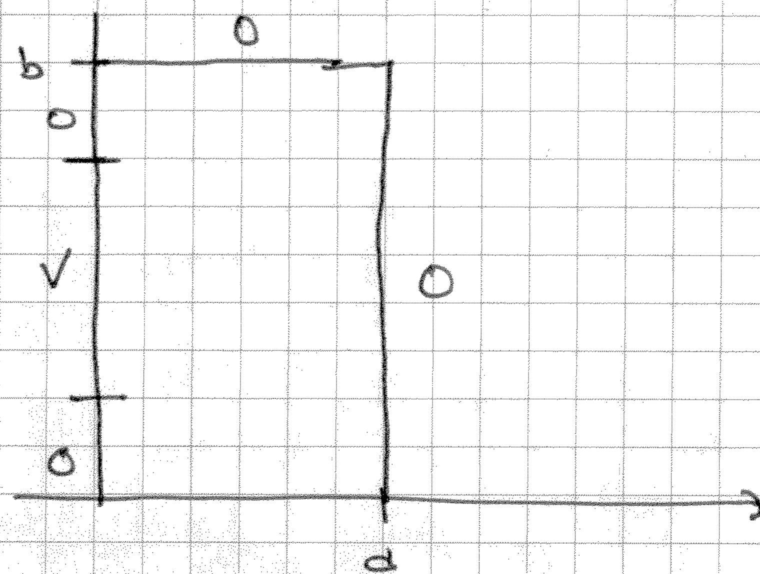
$$= \frac{\mu_0 \pi a^2 N I_0 A}{l \tau \rho 2\pi a}$$

$$= \frac{\mu_0 a A N I_0}{2 l \tau \rho}$$

$$= \frac{\left(4\pi \times 10^{-7} \frac{Tm}{A}\right)(0.01m)(1 \times 10^{-6} m^2)(100)\left(\frac{1}{4} A\right)}{2(0.2m)(1 \times 10^{-3} s)(2.4 \times 10^{-8} \Omega m)}$$

$= 0.033\,A$ no worries.

4.5



Laplace's Eqn $\nabla^2 V = 0$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Separate Solution

$$V = XY$$

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{= +k^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{= -k^2} = 0$$

Want Y solution to oscillate

General Solution

$$V = (\sin k\cancel{x}y, \cos ky) \times (e^{kx}, e^{-kx})$$

Boundary Conditions

$$V(\cancel{x},\cancel{0}) = 0 \implies \text{no } \cos ky \text{ term}$$

$$V(x,b) = 0 \implies \sin kb = 0$$

$$kb = n\pi$$

$$k_n = \frac{n\pi}{b}$$

General Solution

$$V(x,y) = \sum_n \left(A_n e^{k_n x} + B_n e^{-k_n x}\right) \sin k_n y$$

$$V(a,y) = 0 = \sum_n \left(A_n e^{k_n a} + B_n e^{-k_n a}\right) \sin k_n y$$

$\implies$ $\sin k_n y$ orthogonal, so ~~zero~~ term by term

$$A_n e^{k_n a} + B_n e^{-k_n a} = 0$$

$$B_n = -A_n e^{2k_n a}$$

Final Boundary Condition

$$V(0,y) = \begin{cases} 0 & y < \frac{b}{4} \\ V_0 & b/4 < y < \frac{3b}{4} \\ 0 & y > \frac{3b}{4} \end{cases}$$

$$= \sum_n (A_n + B_n) \sin k_n y$$

Multiply by $\sin k_m y$ and integrate

$$\int_0^b dy \sin k_m y \, V(0,y) = \sum_n (A_n + B_n) \int_0^b \sin k_m y \sin k_n y \, dy$$

$$= \frac{b}{2} (A_m + B_m)$$

$$= \int_{b/4}^{3b/4} V_0 \sin \frac{m\pi y}{b} \, dy$$

$$= \frac{V_0 b}{m\pi} \cos \frac{m\pi y}{b} \Big|_{b/4}^{3b/4}$$

$$= \frac{V_0 b}{m\pi}\left(\cos\frac{3m\pi}{4} - \cos\frac{m\pi}{4}\right)$$

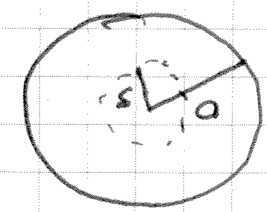
$$= \frac{b}{2}\left(A_m + B_m\right) = \frac{b}{2}\left(A_m - A_m e^{2k_m a}\right)$$

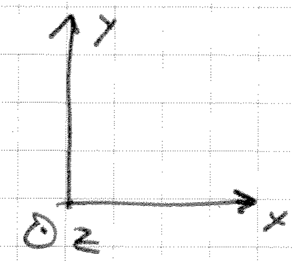
$$A_m = \frac{2V_0}{m\pi}\,\frac{\left(\cos\frac{3m\pi}{4} - \cos\frac{m\pi}{4}\right)}{1 - e^{2k_m a}}$$

$$B_m = -A_m e^{2k_m a}$$

$$V(x,y) = \sum_n \left(A_n e^{k_n x} + B_n e^{-k_n y}\right)\sin k_n y$$

4.6





$$J_I = \frac{I}{\pi a^2}\hat{z} \qquad \vec{J}_{II} = \frac{J_0}{s} e^{-s/b}\,\hat{z}$$

Compute $\vec{H}_I$ For Amperian path of radius s,

$$I_{fenc} = \pi s^2 J_I = I\frac{s^2}{a^2}$$

$$\oint \vec{H}_I \cdot d\vec{l} = I_{fenc} = 2\pi s H_I$$

$$\vec{H}_I = \frac{I s^2/a^2}{2\pi s} \text{ ccw}$$

$$= \frac{I s}{2\pi a^2}\hat{\phi}$$

The magnetic field is

$$\vec{H}_I = \frac{\vec{B}_I}{\mu_0 \mu_r} \qquad \vec{B}_I = \frac{\mu_0 \mu_r I s}{2\pi a^2}\hat{\phi}$$

Outside the wire, $s > a$.

$$I_{fenc} = I + \int_0^{2\pi}\int_a^s J_{II}\, da \qquad da = s\,ds\,d\phi$$

$$= I + \int_0^{2\pi} d\phi \int_a^s s\,ds\, \frac{e^{-s/b}}{s} J_0$$

$$= I + 2\pi J_0 \int_a^s e^{-s/b}\, ds$$

$$= I - 2\pi J_0\, b\, e^{-s/b} \Big|_a^s$$

$$= I + 2\pi J_0 b \left(e^{-a/b} - e^{-s/b}\right)$$

$$\vec{H}_{II} = \frac{I_{fenc}}{2\pi s}\hat{\phi} = \frac{I + 2\pi J_0 b \left(e^{-a/b} - e^{-s/b}\right.}{2\pi s}$$

$$\vec{B}_{II} = \mu_0 \vec{H}_{II}$$