Electricity and Magnetism - Test 1 - Spring 2013

Work four of the six problems. Place the problems in the order you wish them graded. The first two problems form the first half test: the second two problems form the second half test.

Problem 1.1 Consider the following electromagnetic field

$$\vec{E}(x, y, z) = E_0 \sin(kx - \omega t)\hat{y}$$

$$\vec{B}(x, y, z) = B_0 \sin(kx - \omega t)\hat{z}$$

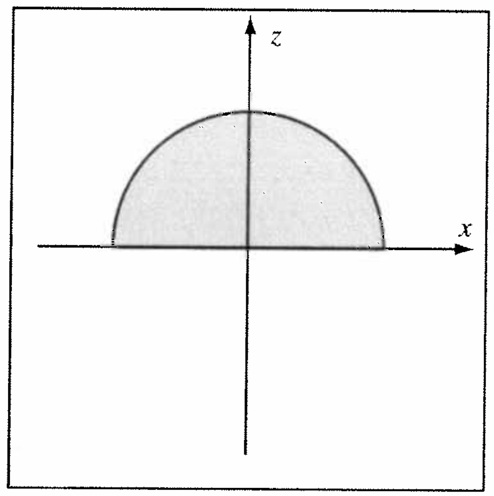
where E_0, B_0, k, and ω are constants which may be related. The field exists in a region of space where both the charge density ρ and the current density $\vec{J}$ are zero. Does this field satisfy Maxwell's equations? If it does not, state all of the equations that are not satisfied for any non-zero choice of E_0, B_0, k, and ω. If the fields do satisfy Maxwell's equations, what algebraic equations must be satisfied by E_0, B_0, k, and ω for Maxwell's equations to be satisfied?

Problem 1.2 As a first model for the charge density of a semiconductor diode we could use

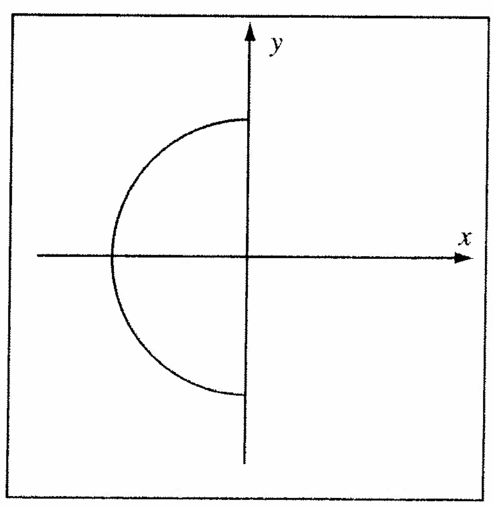
$$\rho(x) = \frac{\rho_0}{a} x \exp(-(x/a)^2).$$

Sketch this charge density and the electric field. Calculate the electric field everywhere.

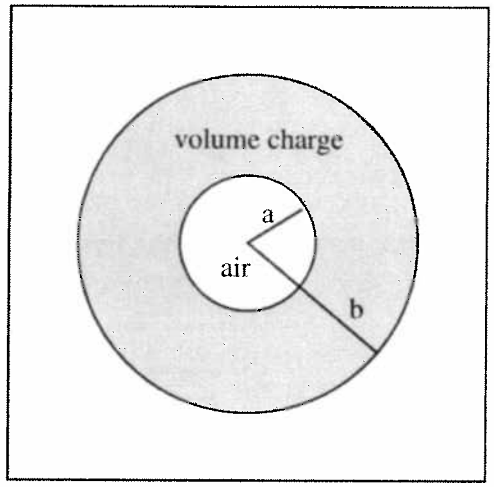
Problem 1.3 Compute the electric field at the origin of a uniformly charged half-sphere with charge density ρ where ρ is non-zero at points $r < a$ and $z > 0$.



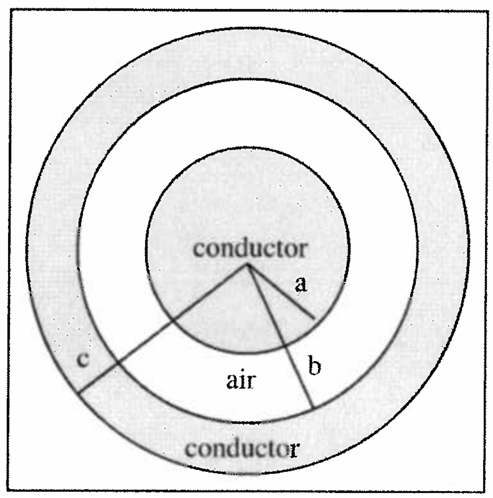
Problem 1.4 Compute the electric potential at a point P along the z axis, $\vec{r}_P = (0, 0, z)$, of a half-circle with constant linear charge density λ. The half-circle lies in the $x - y$ plane, has radius a, and is non-zero for $\pi/2 < \phi < 3\pi/2$.



Problem 1.5 A NON-UNIFORM spherical volume charge has charge density $\rho(r) = \gamma r$ for $a < r < b$ and zero otherwise; γ is a constant. Calculate the electric field and electric potential everywhere. Let $V(\infty) = 0$.



Problem 1.6 A capacitor is formed of two concentric spherical conductors. The inner conductor has radius a and the outer conductor has inner radius b and outer radius c. Compute the capacitance between the inner and outer conductors. Compute the energy stored in the capacitor if the inner conductor has charge Q and the outer conductor charge $-Q$ in two ways: (1) from the capacitance, (2) from the energy density of the fields.



1.1

Gauss $\nabla \cdot \vec{E} = 0$

$$\nabla \cdot \vec{E} = \frac{\partial E}{\partial y} = 0 \quad \checkmark$$

No Mag Monopoles

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{B} = \frac{\partial B}{\partial y} = 0 \quad \checkmark$$

Faraday

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{E} = \frac{\partial V_y}{\partial x} = E_0 k \cos(kx - \omega t)\, \hat{z}$$

$$\frac{\partial \vec{B}}{\partial t} = -B_0 \omega \cos(kx - \omega t)\, \hat{z}$$

Satified if $E_0 k = B_0 \omega$

Ampere

$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{B} = -\frac{\partial}{\partial x} B_0 \sin(kx - \omega t)\,\hat{y}$$

$$= -k B_0 \cos(kx - \omega t)\,\hat{y}$$

$$\frac{\partial \vec{E}}{\partial t} = -\omega E_0 \cos(kx - \omega t)\,\hat{y}$$

Satisfied if

$$-k B_0 = -\mu_0 \varepsilon_0 \omega E_0$$

So we can satisfy Maxwell's eqns if

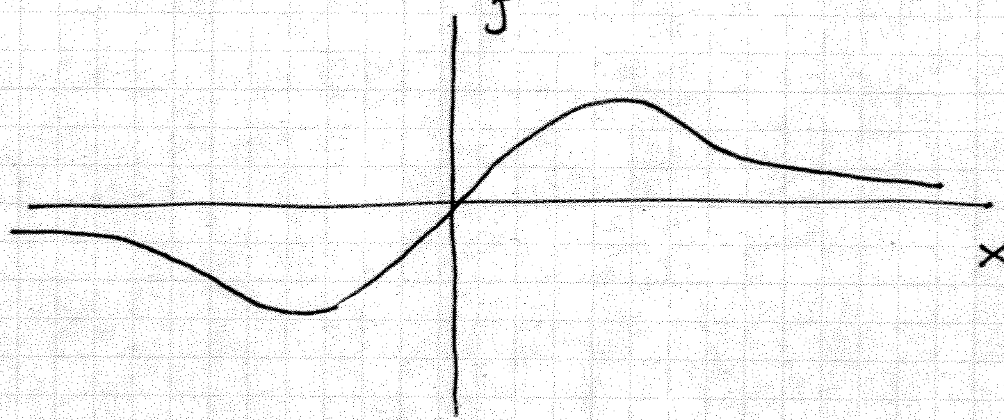
$$E_0 k = B_0 \omega$$

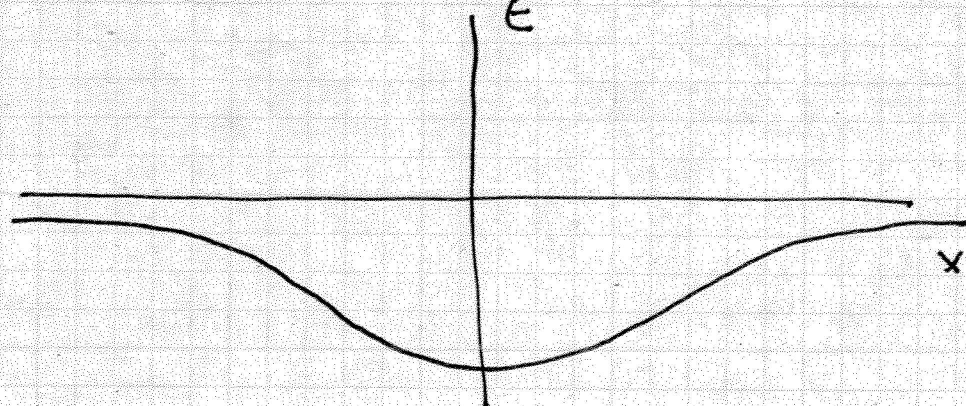
$$B_0 k = \mu_0 \varepsilon_0 \omega E_0$$

Solve $\quad B_0 k = \mu_0 \varepsilon_0 \omega \left(\frac{B_0 \omega}{k} \right)$

$$\frac{1}{\mu_0 \varepsilon_0} = \frac{\omega^2}{k^2} \qquad \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \text{speed of light}$$

1.2





Use Gauss' Law

A cylindrical Gaussian surface that encloses all charge, encloses zero net charge, so the field at $\pm\infty$ is zero.

Now use a Gaussian surface with one end at $-\infty$ and the other end at x. The charge enclosed if the surface has end area A

is $Q_{enc} = \int \rho \, d\tau = \int \rho A \, dx$

$$Q_{enc} = \frac{\rho_0}{a} A \int_{-\infty}^{x} x e^{-(x/a)^2} dx$$

$$u = (x/a)^2 \qquad du = \frac{2}{a^2} x\, dx$$

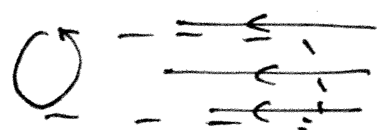
$$Q_{enc} = \frac{\rho_0 A}{a} \cdot \frac{a^2}{2} \cdot \int_{\infty}^{\left(\frac{x}{a}\right)^2} e^{-u} du$$

$$= \frac{\rho_0 A a}{2} \left[-e^{-u} \right]_{\infty}^{\left(\frac{x}{a}\right)^2}$$

$$= -\frac{\rho_0 A a}{2} e^{-(x/a)^2}$$

Both units and directions check.

<u>Gauss Law</u>



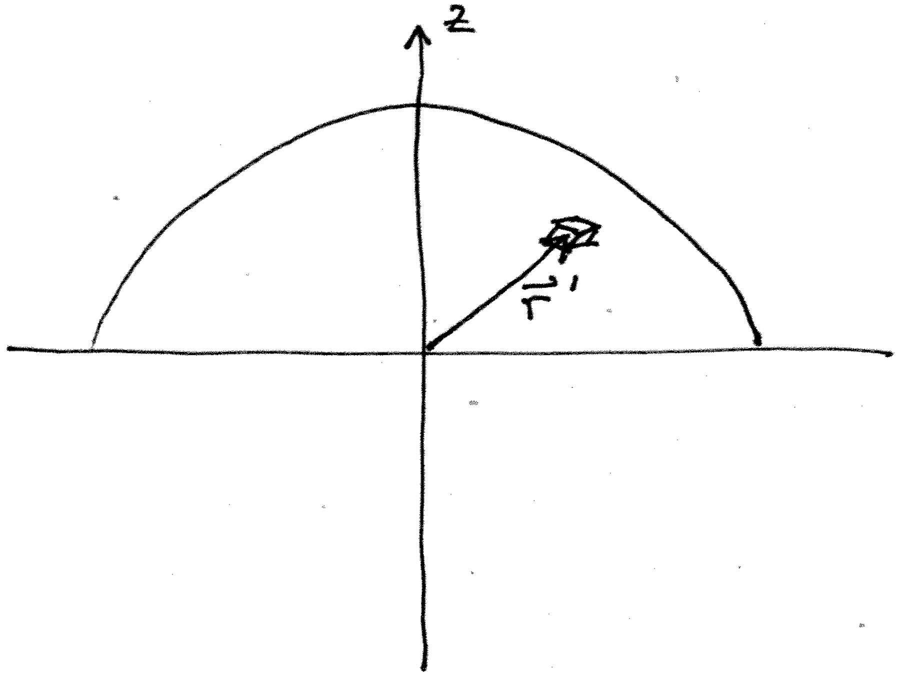
$$\phi = E(x) A - E(-\infty) A$$

$$= \frac{Q_{enc}}{\varepsilon_0} \qquad E(-\infty) = 0$$

$$E(x) = \frac{Q_{enc}}{A \varepsilon_0} = \cancel{-\rho_0 A a}$$

$$= -\frac{\rho_0 a}{2\varepsilon_0} e^{-(x/a)^2} \hat{x}$$

(1.3) Use spherical coordinates



Source Point $\quad \vec{r}\,' = r'\,\hat{r}\,'$

Field Point $\quad \vec{r} = 0$

Displacement Vector $\quad \vec{r}\,'' = \vec{r} - \vec{r}\,' = -r'\,\hat{r}\,'$

Length $\quad r'$

Unit Vector $\quad -\hat{r}\,'$

<u>Coulomb's Law</u>

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\, d\tau'\,\hat{r}\,''}{(r'')^2}$$

$$d\tau' = (dr')(r'd\theta')(r'\sin\theta' d\phi')$$

$$\vec{E} = \frac{-\rho}{4\pi\varepsilon_0} \int \frac{dr'd\theta'd\phi' r'^2 \sin\theta'}{(r')^2} \hat{r}'$$

The field must point in the $-\hat{z}$ direction by symmetry. Write $\hat{r}'$ in cartesian and throw away $\hat{x}, \hat{y}$ term

<u>Griffith's Cover</u>

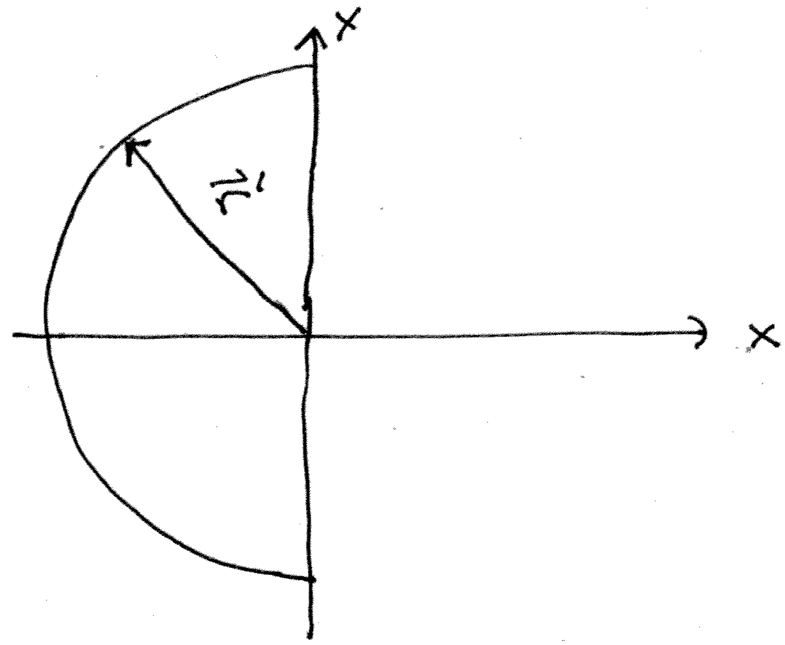
$$\hat{r}' = \underbrace{\sin\theta'\cos\phi'\hat{x} + \sin\theta'\sin\phi'\hat{y}}_{\text{integrates to zero}} + \cos\theta'\hat{z}$$

$$\vec{E} = \frac{-\rho_0\hat{z}}{4\pi\varepsilon_0} \underbrace{\int_0^a dr'}_{a} \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} \underbrace{\int_0^{\pi/2} d\theta' \sin\theta' \cos\theta'}_{\frac{1}{2}}$$

$$= \frac{-\rho_0}{4\pi\varepsilon_0} \cdot \pi a \, \hat{z} = -\frac{\rho_0 a}{4\varepsilon_0}\hat{z}$$

Units and direction check.

1.4



<u>Field point</u> $\vec{r} = (0,0,z)$

<u>Source point</u> $\vec{r}\,' = s'\hat{s}'$

<u>Displacement</u> $\vec{r}\,'' = \vec{r} - \vec{r}\,' = z\hat{z} - s'\hat{s}'$

<u>Length</u> $r'' = \sqrt{z^2 + s'^2} = \sqrt{z^2 + a^2}$

<u>Potential</u>

$$V(z) = \frac{1}{4\pi\epsilon_0} \int \frac{\lambda\, d\ell'}{r''} \qquad d\ell' = a\,d\phi'$$

$$= \frac{\lambda a}{4\pi\epsilon_0} \int_{\pi/2}^{\frac{3\pi}{2}} \frac{d\phi'}{\sqrt{z^2+a^2}}$$

$$= \frac{\lambda a}{4\pi\epsilon_0 \sqrt{z^2+a^2}} \cdot \pi = \frac{\lambda a}{4\epsilon_0\sqrt{z^2+a^2}}$$

Units Check

1.5

<u>Region I</u> $(r<a)$ $\quad Q_{enc} = 0 \quad \vec{E}_I = 0$

<u>Region II</u> $(a<r<b)$

$$Q_{enc} = \int_a^r 4\pi r^2 \rho \, dr$$

$$= 4\pi\gamma \int_a^r r^3 dr$$

$$= 4\pi\gamma \frac{r^4}{4}\Big|_a^r = \pi\gamma(r^4 - a^4)$$

<u>Field by Gauss</u>

$$\phi = 4\pi r^2 E = Q_{enc}/\varepsilon_0$$

$$\vec{E}_{II} = \frac{\pi\gamma(r^4-a^4)}{4\pi r^2 \varepsilon_0}\hat{r} = \frac{\gamma(r^4-a^4)}{4\varepsilon_0 r^2}\hat{r}$$

<u>Region III</u> $\quad Q_{enc} = \pi\gamma(b^4 - a^4)$

$$\vec{E}_{III} = \frac{\pi\gamma(b^4-a^4)}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{\gamma(b^4-a^4)}{4\varepsilon_0 r^2}\hat{r}$$

Potential In region III we must have the potential of a point charge

$$V_{III} = \frac{\pi\gamma(b^4-a^4)}{4\pi\varepsilon_0 r} = \frac{Q}{4\pi\varepsilon_0 r}$$

$$= \frac{\gamma(b^4-a^4)}{4\varepsilon_0 r}$$

Region II

$$V_{II}(r) = -\int E_{II}\,dr$$

$$= -\int \frac{\gamma r^2}{4\varepsilon_0}dr + \int \frac{\gamma a^4}{4\varepsilon_0 r^2}dr$$

$$= \frac{-\gamma r^3}{12\varepsilon_0} - \frac{\gamma a^4}{4\varepsilon_0 r} + C$$

Continuity

$$V_{II}(b) = V_{III}(b)$$

$$\frac{\gamma(b^4-a^4)}{4\varepsilon_0 b} = \frac{-\gamma b^3}{12\varepsilon_0} - \frac{\gamma a^4}{4\varepsilon_0 b} + C$$

$$\frac{\gamma b^3}{4\varepsilon_0} = \frac{-\gamma b^3}{12\varepsilon_0} + C$$

$$C = \frac{1}{3}\frac{\gamma b^3}{\varepsilon_0}$$

Region I There is no field, so V_I constant

Continuity

$$V_I(a) = V_{II}(a) = -\frac{\gamma a^3}{12\varepsilon_0} - \frac{\gamma a^4}{4\varepsilon_0 a} + \frac{1}{3}\frac{\gamma b^3}{\varepsilon_0}$$

$$= \frac{1}{3}\frac{\gamma}{\varepsilon_0}(b^3 - a^3)$$

(1.6) Place $+Q$ on inner conductor, $-Q$ on outer conductor.

The field between the conductors is $\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$

The potential difference is

$$\Delta V_{ab} = -\int_a^b \vec{E}\cdot d\vec{l} \qquad d\vec{l} = \hat{r}\,dr$$

$$= \frac{-Q}{4\pi\epsilon_0}\int_a^b \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0 r}\Big|_a^b$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right) < 0$$

Capacitance

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$= \frac{4\pi\epsilon_0}{\frac{1}{a} - \frac{1}{b}} = \frac{4\pi\epsilon_0(ab)}{b-a}$$

Energy

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\cdot\frac{Q^2(b-a)}{4\pi\varepsilon_0 ab}$$

$$= \frac{Q^2(b-a)}{8\pi\varepsilon_0 ab}$$

Energy Density of Field

$$u = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 \frac{Q^2}{16\varepsilon_0 \pi^2 r^4}$$

$$= \frac{Q^2}{32\pi^2\varepsilon_0}\cdot\frac{1}{r^4}$$

Integrate energy density

$$U = \int u\, d\tau$$

$$= \int_a^b 4\pi r^2 u\, dr = 4\pi\cdot\frac{Q^2}{32\pi^2\varepsilon_0}\int_a^b \frac{dr}{r^2}$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{Q^2(b-a)}{8\pi\varepsilon_0 ab}$$