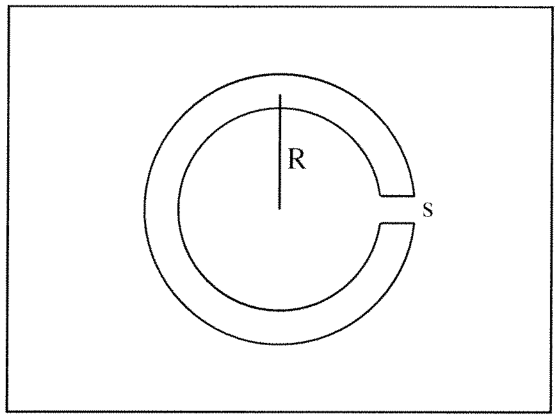
Electricity and Magnetism - Test 3 - Spring 2013

Work four of the six problems. Place the problems in the order you wish them graded. The first two problems form the first half test; the second two problems form the second half test.

Problem 3.1 Consider the possible vector potential $\vec{A} = \gamma x^2 \hat{y} - \gamma y^2 \hat{x}$, where γ is a constant, which I just made up. It is divergenceless, so it is a possible candidate for a vector potential. Find the magnetic field resulting from this potential. Could the magnetic field be a possible magnetostatic field? If not, why not? If it can, what current density produced the field? Is the current density a possible magnetostatic current density?

Problem 3.2 The figure below shows a permanent horseshoe magnet. The magnet is of radius R and has a small gap of width s. The magnet is made of a material with constant magnetization density M_0 which points in the clockwise direction. Compute the magnetic field in the empty gap. To preserve the strength of the magnet, the magnet is stored with a "keeper" filling the gap. The keeper is a piece of iron with relative permeability μ_r that completely fills the gap. Compute the magnetic field in the iron keeper if it is inserted in the gap.

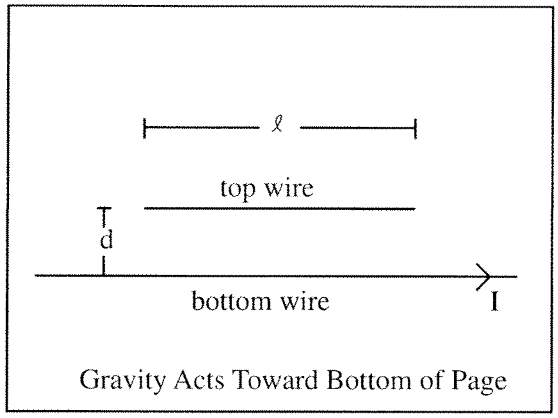


Problem 3.3 A disk with charge density $\sigma(s) = \sigma_0 s$ and radius a is centered at the origin in the $x - y$ plane. The disk is rotated about the z axis with an angular velocity ω. Calculate the magnetic field at the origin.

Problem 3.4 A finite cylinder of radius a and length ℓ is co-axial with the z axis. A surface current density $\vec{K} = K_0 \hat{z}$ flows down the surface of the cylinder. Compute the vector potential at the center of the cylinder. Use this point as the origin of your coordinate system in the your calculation.

Problem 3.5 An infinitely long, hollow, cylindrical conductor with inner radius a and outer radius b co-axial with the z axis carries a current density $\vec{J} = \frac{J_0 a}{s} \hat{z}$. Note, no current flows in the region $s < a$. The conductor is surrounded by a linear magnetic material of inner radius b and outer radius c with relative permeability μ_r. The magnetic response of the conductor is negligible ($\mu_r = 1$) and may be ignored. Compute $\vec{H}$ and $\vec{B}$ everywhere.

Problem 3.6 A very long wire, the bottom wire, carries current $I_b = 2\text{A}$ along the x axis. A second wire with mass $m = 0.1\text{kg}$ floats above the first wire, parallel to the first wire, due to the upward magnetic force balancing a downward gravitational force. The second wire is $\ell = 10\text{cm}$ long. The center-to-center spacing between the wires is $d = 1\text{cm}$. How much current, I_t, must the top wire carry and in what direction for it to float with the separation given. The system is drawn below. The current calculated may be either be a very small or a very large number.



3.1 Compute the magnetic field

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\gamma y^2 & \gamma x^2 & 0 \end{vmatrix} = \vec{B}$$

$$\vec{B} = 2\gamma x + 2\gamma y = 2\gamma (x + y)\,\hat{z}$$

<u>Check Maxwell</u>

$$\nabla \cdot \vec{B} = 0 \quad \checkmark$$

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & 2\gamma(x+y) \end{vmatrix}$$

$$= 2\gamma\hat{x} - 2\gamma\hat{y} = \mu_0 \vec{J}$$

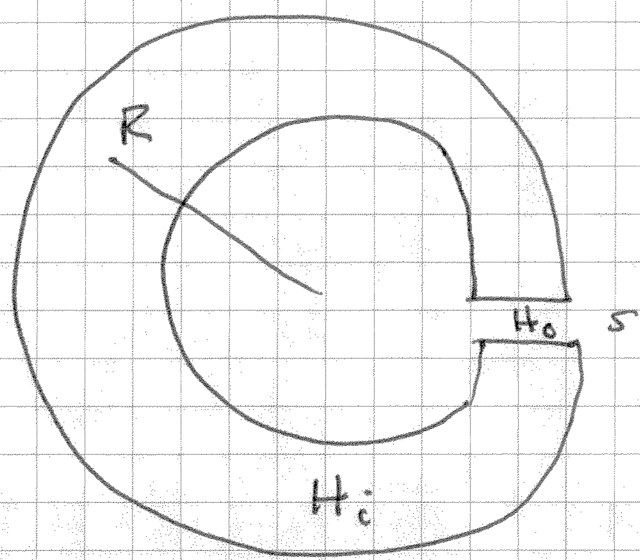
$$\vec{J} = \frac{2\gamma}{\mu_0}(\hat{x} - \hat{y})$$

Check $\nabla \cdot \vec{J}$

$$\nabla \cdot \vec{J} = 0 \quad \checkmark$$

Could be magnetostatic.

3.2



$$mmf = \int_C \vec{H}\cdot d\vec{l} = (2\pi R - s)H_i + sH_o = 0$$

By $\nabla\cdot\vec{B} = 0$, $B_i = B_o$

Outside $\mu_o H_o = B_o$

Inside $\vec{H}_i = \dfrac{\vec{B}_i}{\mu_o} - \vec{M}$

$$H_i = \frac{B_o}{\mu_o} - M_o$$

Substitute

$$(2\pi R - s)\left(\frac{B_o}{\mu_o} - M_o\right) + s\frac{B_o}{\mu_o} = 0$$

$$2\pi R\, B_0 \;-\; 2\pi R\,\mu_0 M_0 \;-\; sB_0 \;+\; s\mu_0 M_0 \;+\; sB_0 = 0$$

$$2\pi R\, B_0 = \mu_0 M_0 (2\pi R - s)$$

$$B_0 = \mu_0 M_0 \left(1 - \frac{s}{2\pi R}\right)$$

If the gap is filled with a magnetic material with permeability μ_r then

$$\mu_0 \mu_r H_0 = B_0$$

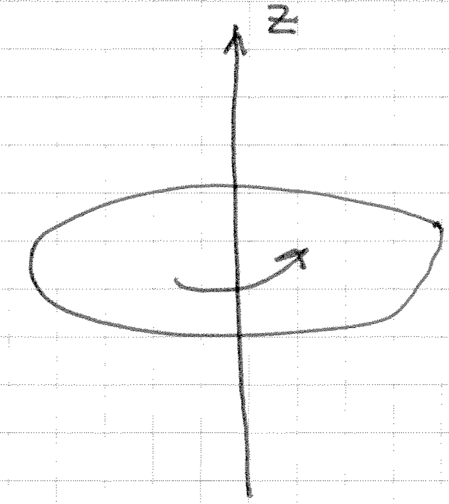
$$(2\pi R - s)\left(\frac{B_0}{\mu_0} - M_0\right) + \frac{sB_0}{\mu_0 \mu_r} = 0$$

$$(2\pi R - s)\frac{B_0}{\mu_0} + \frac{sB_0}{\mu_0\mu_r} = M_0(2\pi R - s)$$

$$B_0\left(2\pi R - s + \frac{s}{\mu_r}\right) = \mu_0 M_0 (2\pi R - s)$$

$$B_0 = \frac{\mu_0 M_0 (2\pi R - s)}{2\pi R - \left(1 - \frac{1}{\mu_r}\right)s}$$

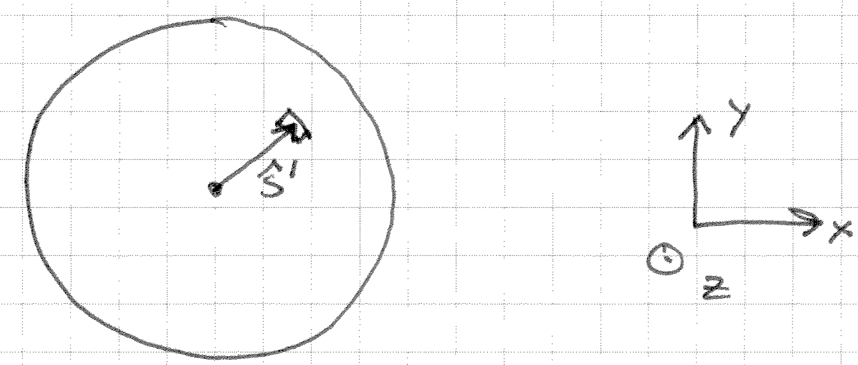
3.3



$$\vec{K} = \sigma \vec{v} = \sigma \omega s \hat{\phi}$$

$$= \sigma_0 \omega s'^2 \hat{\phi}'$$

Biot-Savart



$$\vec{B}(0) = \frac{\mu_0}{4\pi} \int \frac{\vec{K} \times \hat{r}'}{(r'')^2} da'$$

$\vec{r} = 0$

$\vec{r}' = s' \hat{s}'$

$\vec{r}'' = -s' \hat{s}'$

$$da' = s' ds' d\phi'$$

$$\hat{r}' = -\hat{s}'$$

$$\vec{K} \times \hat{r}' = \sigma_0 \omega s'^2 \hat{\phi}' \times (-\hat{s}')$$

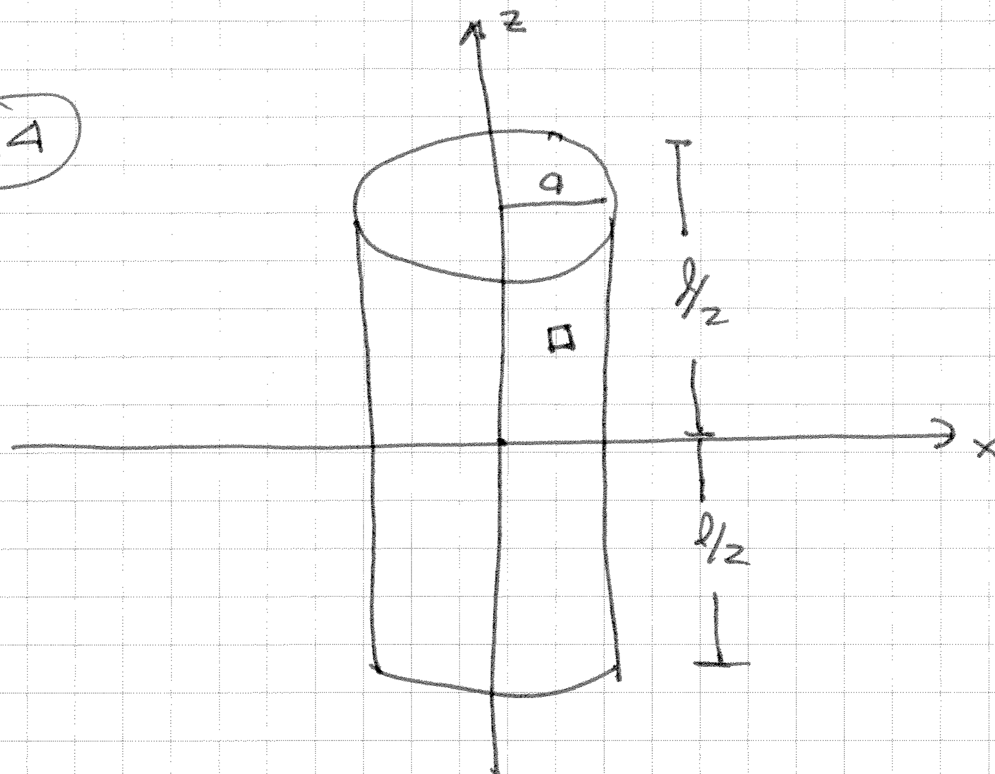
$$= \sigma_0 \omega s'^2 \hat{z}$$

$$\vec{B}(0) = \frac{\mu_0}{4\pi} \underbrace{\int_0^{2\pi} d\phi'}_{= 2\pi} \int_0^a ds' \frac{(\sigma_0 \omega s'^2 \hat{z})\, s'}{(s')^2}$$

$$= \frac{\mu_0 \sigma_0 \omega \hat{z} \cdot 2\pi}{4\pi} \int_0^a s'\, ds'$$

$$= \frac{\mu_0 \sigma_0 \omega \hat{z}}{2} \frac{a^2}{2} = \frac{\mu_0 \sigma_0 \omega a^2}{4} \hat{z}$$

3.4



$$\vec{r}\,' = s'\hat{s}' + z'\hat{z}$$

$$\vec{r} = 0$$

$$\vec{r}'' = -\hat{s}'s' - z'\hat{z} \qquad r'' = \sqrt{s'^2 + z'^2}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}'\,da'}{r''}$$

Since the current is confined to the surface,

$$s' = a \,, \quad r'' = \sqrt{a^2 + z'^2}$$

$$da' = a\,d\phi'\,dz'$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{(K_0\hat{z})(a\,d\phi'\,dz')}{\sqrt{a^2+z'^2}}$$

$$\vec{A} = \frac{\mu_0 K_0 \hat{z}\, a}{4\pi} \int_{-l/2}^{l/2} dz' \int_0^{2\pi} d\phi' \frac{1}{\sqrt{a^2+z'^2}}$$

$$= \mu_0 K_0 a \hat{z} \int_0^{l/2} \frac{dz'}{\sqrt{a^2+z'^2}}$$

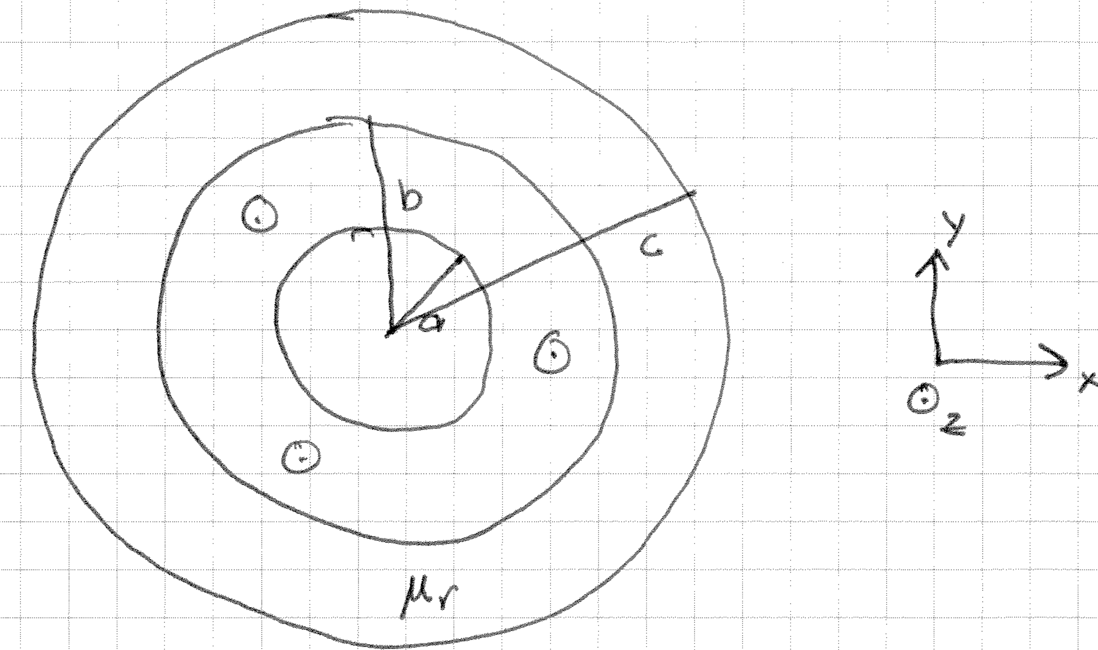
<u>Wolfram</u>

$$\int \frac{dz}{\sqrt{a^2+z^2}} = \ln\left(z + \sqrt{a^2+z^2}\right) + C$$

$$\vec{A} = \mu_0 K_0 a \hat{z} \left(\ln\left(l/2 + \sqrt{a^2+(l/2)^2} \right. - \ln(a) \right)$$

$$= \mu_0 K_0 a \hat{z} \ln\left(\frac{l}{2a} + \sqrt{1+\left(\frac{l}{2a}\right)^2} \right)$$

3.5



$$\oint \vec{H} \cdot d\vec{l} = I_{fenc}$$

Region I $(s<a)$

$$I_{fenc} = 0 \Rightarrow \vec{B}_I = 0 \,, \quad \vec{H}_I = 0$$

Region II $(a<s<b)$

$$I_{fenc} = \int_S \vec{J} \cdot d\vec{a}$$

$$d\vec{a} = \hat{z} \, s \, ds \, d\phi$$

$$I_{fenc} = \int_0^{2\pi} d\phi \int_a^{s} s\,ds \cdot \frac{J_0 a}{s}$$

$$= J_0 a \int_0^{2\pi} d\phi \int_a^{s} ds$$

$$= 2\pi J_0 a (s - a)$$

$$\oint \vec{H}_{II} \cdot d\vec{l} = 2\pi s H_{II} = I_{fenc} = 2\pi J_0 a (s - a)$$

$$\vec{H}_{II} = J_0 a \left(1 - \frac{a}{s}\right) \text{ ccw}$$

Right hand rule.

Since there is no ~~del~~ magnetic material in Region II,

$$\vec{B}_{II} = \mu_0 \vec{H}_{II}$$

$$= \mu_0 J_0 a \left(1 - \frac{a}{s}\right) \text{ ccw}$$

~~Region III (b < s < c)~~

$$\cancel{I_{fenc} = 2\pi J_0 a \left(1 - \frac{a}{b}\right)}$$

$$\cancel{\vec{H}_{III} = I_{fenc}}$$

Region III $(b < s < c)$

$$I_{fenc} = 2\pi J_0 a (b - a)$$

$$\vec{H}_{III} = \frac{I_{fenc}}{2\pi s} \text{ ccw}$$

$$= \frac{J_0 a (b - a)}{s} \text{ ccw} \quad = \vec{H}_{IV}$$

$$\vec{B}_{III} = \mu_0 \mu_r \vec{H}_{III} = \frac{\mu_0 \mu_r J_0 a (b - a)}{s} \text{ ccw}$$

Region IV $s > c$

$$\vec{H}_{IV} = \frac{J_0 a (b - a)}{s} \text{ ccw}$$

$$\vec{B}_{IV} = \mu_0 \vec{H}_{IV} = \frac{\mu_0 J_0 a (b - a)}{s} \text{ ccw}$$

3.6

Opposite currents repel, so the current in the top wire must be to the top of the page.

The field of the bottom wire at the top wire is

$$B = \frac{\mu_0 I_b}{2\pi d}$$

The force on the top wire is

$$F = I_t \ell B = \frac{\mu_0 \ell I_b I_t}{2\pi d} = mg$$

The current that must flow in the top wire to balance the force of gravity is

$$I_t = \frac{2\pi d m g}{\mu_0 \ell I_b} = \frac{(2\pi)(0.01\text{m})(0.1\text{kg})(9.81\text{m/s}^2)}{(4\pi\times10^{-7}\frac{\text{Tm}}{\text{A}})(0.1\text{m})(2A)}$$

$$= 2.45\times10^{5}\text{ A}$$