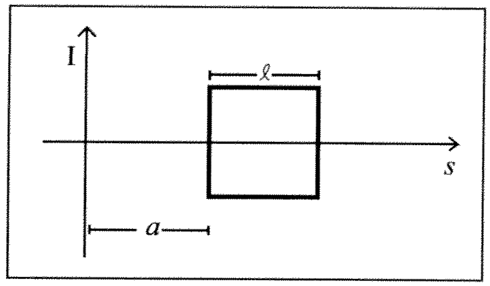
Electricity and Magnetism - Final Exam - Spring 2014

Work four of the six problems. Place the problems in the order you wish them graded. The first two problems form the first half test; the second two problems form the second half test. If you turn in all six problems, then 75% of your score on the last two problems will be used to replace your lowest test score (for better of worse).

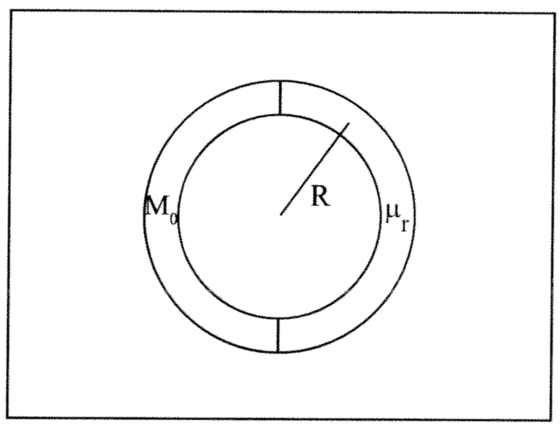
Problem 4.1 A copper pipe has inner radius a and outer radius b. The pipe is a length ℓ long. The conductivity of the copper increases exponentially with ℓ., $\sigma(x) = \sigma_0 \exp(x/\ell)$. Compute the resistance of the pipe.

Problem 4.2 An infinite straight wire carries a time varying current $I(t) = I_0 \sin(\omega t)$. A distance a from a square loop of wire with resistance R and side length ℓ. Both the infinite wire and the loop are in the plane of the page. Compute the current induced in the square loop.



Problem 4.3 A cylindrical region of space of radius a co-axial with the z axis contains a time varying electric field $\vec{E}(t) = E_0 \sin(\omega t)\hat{z}$ where E_0 and ω are constant. Compute the magnetic field in the region.

Problem 4.4 A ring of radius R is composed a permanent magnetic material with magnetization M_0 and a linear magnetic material with relative permeability μ_r. Each occupy half the radius as drawn. Compute the magnetic field in the linear magnetic material.



Problem 4.5 A spherically symmetric system of electric charge has volume charge density $\rho = \gamma r$ for $r < a$ and $\rho = 0$ for $r > a$. The region $r < a$ also contains a linear dielectric with dielectric constant κ. Compute $\vec{D}$ and $\vec{E}$ everywhere.

Problem 4.6 A disk of radius a lies in the $x - y$ plane. The disk has surface charge density γs where γ is a constant. Compute the electric field a distance R along the positive z axis.

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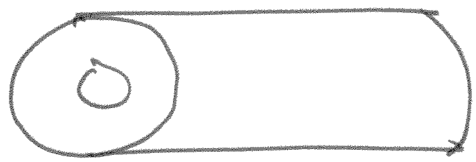


The current I through any cross-section must be constant.

$$J = \frac{I}{\text{Area}} = \frac{I}{\pi(b^2-a^2)}$$

The electric field is $J = \sigma E$

$$E = \frac{I}{\sigma \pi (b^2-a^2)}$$

$$= \frac{I}{\sigma_0 \pi (b^2-a^2)} e^{-\frac{x}{l}}$$

The potential difference is

$$\Delta V = -\int \vec{E}\cdot d\vec{l} = -\frac{I}{\sigma_0 \pi (b^2-a^2)} \int_0^l e^{-x/l}\, dx$$

$$= \frac{I l}{\sigma_0 \pi (b^2-a^2)} e^{-x/l} \Big|_0^l$$

$$|\Delta V| = \frac{I l}{\sigma_0 \pi (b^2-a^2)} \left(1 - e^{-l/l}\right)$$

$$R = \frac{|\Delta V|}{I}$$

$$= \frac{\ell (1-e)}{\sigma_0 \pi (b^2 - a^2)}$$

(2) The field of the infinite wire is

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

Let I point in $+z$ direction

Let the path around loop be clockwise with normal $+\hat{\phi}$.

$d\vec{a} = l\,ds\,\hat{\phi}$

The magnetic flux through the loop is

$$\Phi_m = \int \vec{B}\cdot d\vec{a} = l\int_a^{a+l} B\,ds$$

$$= \frac{\mu_0 I l}{2\pi}\int_a^{a+l}\frac{ds}{s}$$

$$= \frac{\mu_0 I l}{2\pi}\ln\left(\frac{a+l}{a}\right)$$

Faraday's Law says the time rate of change of the flux is the emf and the current is the emf over the resistance

$$I_{\text{induced}} = \frac{\text{emf}}{R} = \cancel{\text{to}} -\frac{1}{R}\frac{d\Phi_m}{dt}$$

$$= -\frac{\mu_0 l}{2\pi R}\ln\left(\frac{a+l}{a}\right)\frac{d}{dt}I_0\sin\omega t$$

$$I_{induced} = \frac{-\mu_0 l I_0 \omega}{2\pi R} \ln\left(\frac{a+l}{a}\right) \cos \omega t$$

(3) The displacement current density is

$$\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \varepsilon_0 E_0 \omega \cos\omega t \,\hat{z}$$

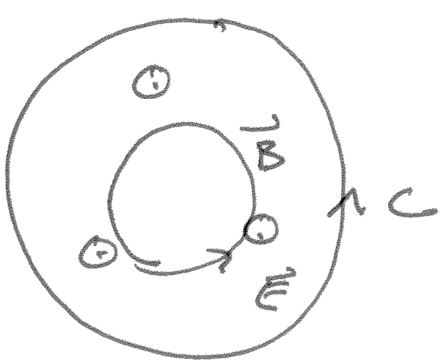
For $s < a$, $(\hat{n} = \hat{z})$

$$I_{d,enc} = \pi s^2 \varepsilon_0 E_0 \omega \cos\omega t$$

$$\oint \vec{B}\cdot d\vec{l} = 2\pi s B = \mu_0 I_{d,enc}$$

$$\vec{B} = \frac{\pi s^2 \varepsilon_0 E_0 \omega \cos\omega t}{2\pi s}\,\hat{\phi}$$

$$= \frac{s\,\varepsilon_0 E_0 \omega \cos\omega t}{2}\,\hat{\phi}$$



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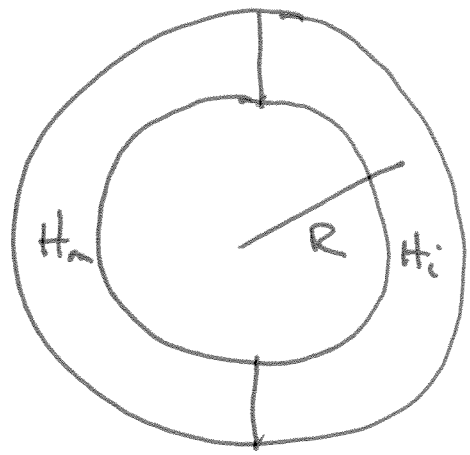

$$\oint \vec{H} \cdot d\vec{l} = 0 = \pi R H_m + \pi R H_i = 0$$

The magnetic field in both materials is equal by no magnetic monopoles.

$$B_i = B_m \equiv B$$

$$H_i = \frac{B_i}{\mu_0 \mu_r} \qquad H_m = \left(\frac{B_m}{\mu_0} - M_o \right)$$

$$\frac{B}{\mu_0} - M_o + \frac{B}{\mu_0 \mu_r} = 0$$

$$\frac{B}{\mu_0} \left(1 + \frac{1}{\mu_r} \right) = M_o$$

$$B = \frac{\mu_0 M_o}{1 + \frac{1}{\mu_r}}$$

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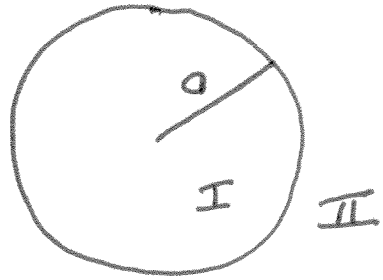

$$\nabla \cdot \vec{D} = \rho_f$$

$$\oint_S \vec{D} \cdot d\vec{a} = \int_V \rho_f \, d\tau = Q_{fenc} = 4\pi r^2 D$$

For $r < a$,

$$Q_{enc} = \int_0^s 4\pi r^2 \rho_f \, dr$$

$$= 4\pi\gamma \int_0^r r^3 \, dr$$

$$= \pi \gamma r^4$$

$$\vec{D}_I = \frac{Q_{fenc}}{4\pi r^2} \hat{r} = \frac{\pi \gamma r^4}{4\pi r^2} \hat{r}$$

$$= \frac{\gamma r}{4} \hat{r}$$

$$\vec{D}_{I} = \kappa \varepsilon_0 \vec{E}_{I}$$

$$\vec{E}_{I} = \frac{\gamma r}{4 \varepsilon_0 \kappa} \hat{r}$$

Region II $\quad s > a$

$$Q_{enc} = \pi \gamma a^4$$

$$\vec{D}_{II} = \frac{Q_{enc}}{4 \pi r^2} \hat{r} = \frac{\gamma a^4}{4 r^2} \hat{r}$$

$$\vec{E}_{II} = \frac{\vec{D}_{II}}{\varepsilon_0} = \frac{\gamma a^4}{4 \varepsilon_0 r^2} \hat{r}$$

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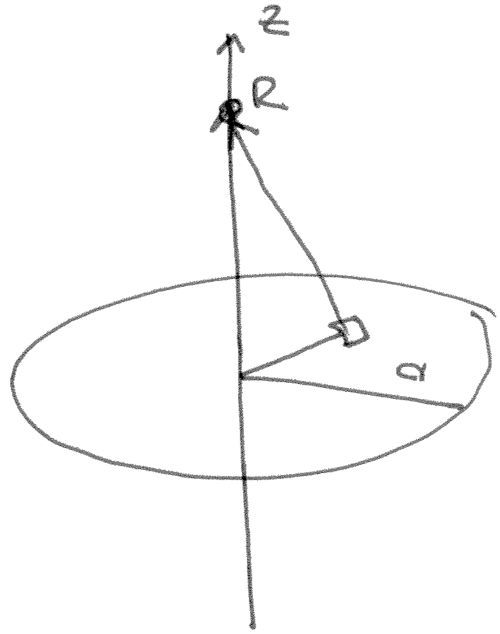

$$\vec{r} = (0,0,R) \qquad \vec{r}\,' = s'\hat{s}'$$

$$\vec{r}\,'' = \vec{r} - \vec{r}\,' = \cancel{s'\hat{s}' - R\hat{z}}$$

$$= R\hat{z} - s'\hat{s}'$$

$$r'' = \sqrt{R^2 + s'^2}$$

$$\vec{E} = \int \frac{k\,\sigma\,\vec{r}\,''\,da'}{(r'')^3} \qquad da' = s'\,ds'\,d\phi'$$

$$= k \int_0^{2\pi} d\phi' \int_0^{a} \frac{\gamma s'\, s'\,ds'}{(s'^2 + R^2)^{3/2}} \left(R\hat{z} - s'\hat{s}'\right)$$

The $\hat{s}'$ integral integrates to zero.

$$\vec{E}(\vec{r}) = \gamma k 2\pi R \hat{z} \int_0^a \frac{s'^3 \, ds'}{(s'^2+R^2)^{3/2}}$$

$$= \gamma k 2\pi R \hat{z} \left(\sqrt{s'^2+R^2} + \frac{R^2}{\sqrt{x^2+R^2}} \right) \Bigg|_0^a$$

$$= \gamma k 2\pi R \hat{z} \left(\sqrt{a^2+R^2} + \frac{R^2}{\sqrt{a^2+R^2}} - R - R \right)$$

$$= \gamma k 2\pi R^2 \hat{z} \left(\frac{\sqrt{a^2+R^2}}{R} + \frac{R}{\sqrt{a^2+R^2}} - 2 \right)$$

$$= 2\pi\gamma k R^2 \hat{z} \left(\frac{a^2+R^2}{R\sqrt{a^2+R^2}} + \frac{R}{\sqrt{a^2+R^2}} - \frac{2\sqrt{a^2+R^2}}{\sqrt{a^2+R^2}} \right)$$

$$= 2\pi\gamma k R \left(\frac{a^2}{R} + R - 2\sqrt{a^2+R^2} \right) \frac{R}{\sqrt{a^2+R^2}} \hat{z}$$

$$= 2\pi\gamma k \left(a^2 + R^2 - 2R\sqrt{a^2+R^2} \right) \frac{R}{\sqrt{a^2+R^2}} \hat{z}$$