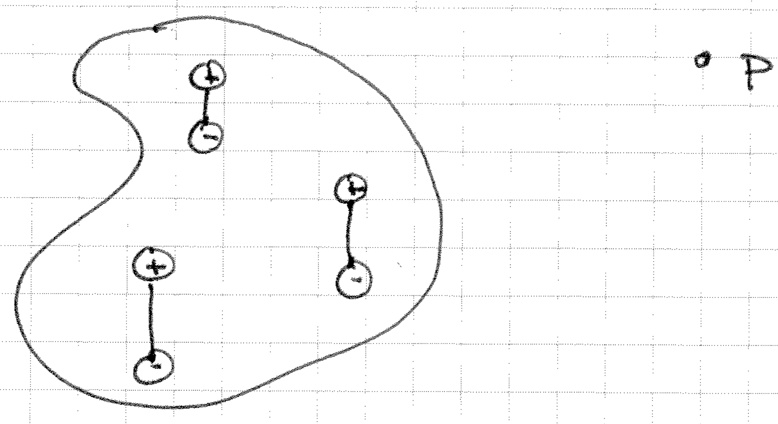
Polarization

Now that we understand the dipole moment we can consider building a charge distribution of a collection of dipoles.



Each dipole produces a potential

$$V_{dip} = \frac{k\vec{p}\cdot\vec{r}}{r^3}$$

and the total potential of the system is

$$V(\vec{r}_p) = \sum_i \frac{k\vec{p}_i\cdot\vec{r}''}{(r'')^3}$$

If the dipoles we small and continuously distributed we could describe the system using a dipole moment per unit volume.

Dfn <u>Polarization</u> ($\vec{P}$) - Dipole moment per unit volume.

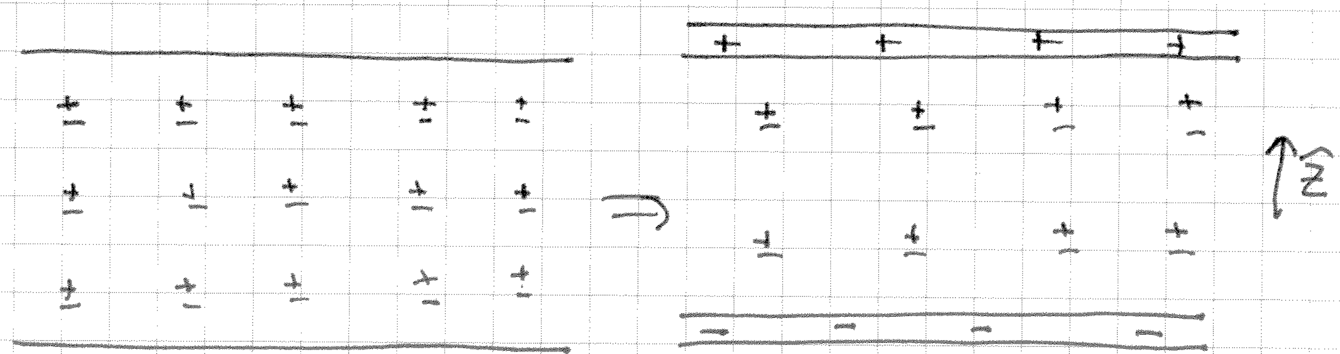
The potential of the system of dipoles is then

$$V(\vec{r}) = \int_V \frac{\vec{P}(\vec{r}\,') \cdot \vec{r}\,''}{(r'')^3}\, d\tau'$$

<u>Let's consider some simple models.</u>

All physical materials contain a positive charge density ρ_+ formed of the protton and an equal but opposite negative charge density ρ_- formed of the electrons.

We can build a dipole moment per unit volume by displacing the center of the protons and the electrons by a distance d.



This displacement of the atomic charge densities creates a thin positive layer of charge on the top surface $\sigma_+ = \rho_+ d$ and a thin layer of negative charge on the bottom surface $\sigma_- = \rho_- d$

The dipole moment per unit volume produced is

$$\vec{P} = \rho_+ d \, \hat{z} = \sigma_+ \hat{z}$$

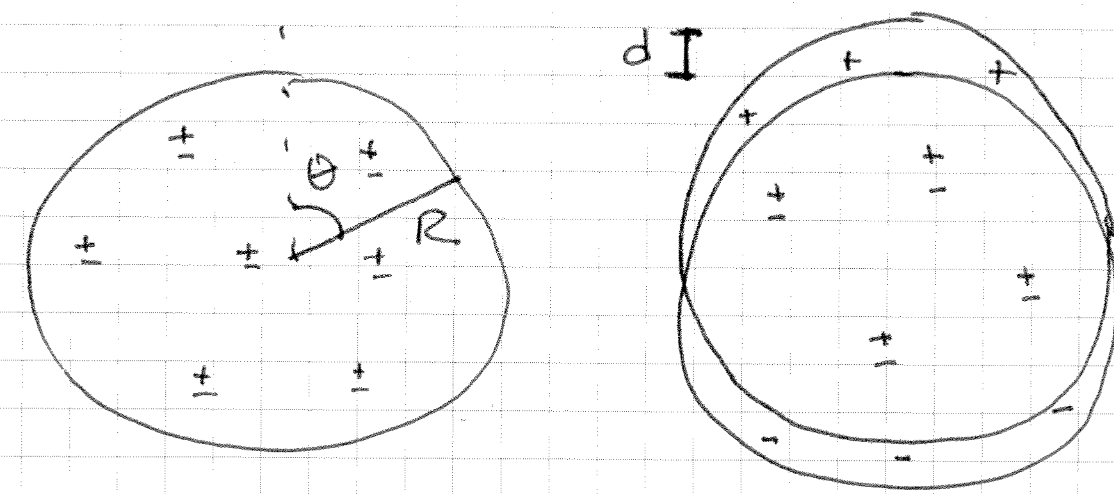
The surface charge density at the top surface is the

$$\sigma_+ = \vec{P} \cdot \hat{n} = \vec{P} \cdot \hat{z}$$

The electric field inside the charge is

$$\vec{E} = -\frac{\sigma_+}{\varepsilon_0} \hat{z} = -\frac{\vec{P}}{\varepsilon_0}$$

Now try the same thing with a sphere



Displace positive charge density ρ_+ by d.

- This polarization produce a surface charge, but no volume charge as before.

- The total dipole moment of the system is

$$\vec{p} = \vec{P} V \quad \text{where } V \text{ is the volume}$$

The polarization is again $\vec{P} = \rho_+ d$

$$\vec{p} = \frac{4}{3}\pi R^3 \rho_+ d \hat{z}$$

$$= Q_+ d \hat{z} \quad \text{as we expected.}$$

The surface charge density produced is

$$\sigma(\theta) = \rho_+ d \cos\theta$$

where $d \cos\theta$ is the thickness of the surface layer.

We can immediately compute the potential ~~field~~ far from the sphere

$$V(\vec{r}) = \frac{\vec{p} \cdot \vec{r}''}{4\pi\varepsilon_0 r''^3}$$

Let's see if we can get the potential everywhere, exactly.

The only physical effect of the polarization that matters in calculating the fields is the net charge produced.

$$\sigma(\theta) = \rho_+ d \cos\theta = \sigma_0 P_1(\cos\theta)$$

$$\sigma_0 \equiv \rho_+ d$$

Solve the boundary value Problem

Inside $r < R$

$$V_i = \sum_{n=0} A_n r^n P_n(\cos\theta)$$

Outside $r > R$

$$V_o = \sum_{n=0} B_n r^{-(n+1)} P_n(\cos\theta)$$

The potential is always continuous

$$V_i(R) = V_o(R)$$

$$\sum_n A_n R^n P_n(\cos\theta) = \sum_n B_n R^{-(n+1)} P_n(\cos\theta)$$

By orthogonality,

$$A_n R^n = B_n R^{-(n+1)}$$

$$B_n = R^{2n+1} A_n$$

Now make sure the charge at the surface is correct.

Electrostatic Boundary Condition

$$\left.\frac{\partial V_o}{\partial r}\right|_R - \left.\frac{\partial V_i}{\partial r}\right|_R = \frac{-\sigma(\theta)}{\varepsilon_0}$$

$$\sum_n -B_n(n+1) R^{-(n+2)} P_n(\cos\theta)$$

$$-\sum_n A_n n R^{(n-1)} P_n(\cos\theta) = \frac{-\sigma}{\varepsilon_0}$$

Use $B_n = A_n R^{2n+1}$

$$\sum -\left(A_n R^{2n+1}\right)(n+1) R^{-(n+2)} P_n(\cos\theta)$$

$$-\sum A_n n R^{(n-1)} P_n(\cos\theta) = \frac{-\sigma}{\varepsilon_0}$$

$$\sum -A_n (2n+1) R^{(n-1)} P_n(\cos\theta) = \frac{-\sigma}{\varepsilon_0}$$

$$= \frac{-\sigma_0}{\varepsilon_0} P_1(\cos\theta)$$

By orthogonality, only A_n, $n=1$ is non-zero, $A_n = 0$, $n \neq 1$.

For $n=1$

$$-A_1(2\cdot 1+1) R^{(1-1)} = \frac{-\sigma_0}{\varepsilon_0}$$

$$3A_1 = \frac{\sigma_0}{\varepsilon_0}$$

$$A_1 = \frac{\sigma_0}{3\varepsilon_0} = \frac{\rho d}{3\varepsilon_0} = \frac{|\vec{P}|}{3\varepsilon_0}$$

Inside Polarized Sphere $(r<R)$

$$V_i(r,\theta) = A_1 r^1 P_1(\cos\theta)$$

$$= A_1 r\cos\theta$$

$$= A_1 z$$

$$= \frac{\sigma_0}{3\varepsilon_0} z = \frac{|\vec{P}|}{3\varepsilon_0} \hat{z}$$

$$\vec{E}_i = -\nabla V = -A_1\hat{z} = -\frac{\sigma}{3\varepsilon_0}\hat{z}$$

$$= -\frac{|\vec{P}|}{3\varepsilon_0}\hat{z}$$

$\Longrightarrow$ Field inside is uniform.

Outside Polarized Sphere $r>R$

$$V_0(r,\theta) = \frac{B_1}{r^2} P_1(\cos\theta)$$

$$= \frac{A_1 R^3}{r^2}\cos\theta$$

$$= \frac{|\vec{P}|}{3\varepsilon_0} R^3 \frac{\cos\theta}{r^2}$$

Let $\vec{P} = P_0 \hat{z}$.

The total dipole moment of the sphere is

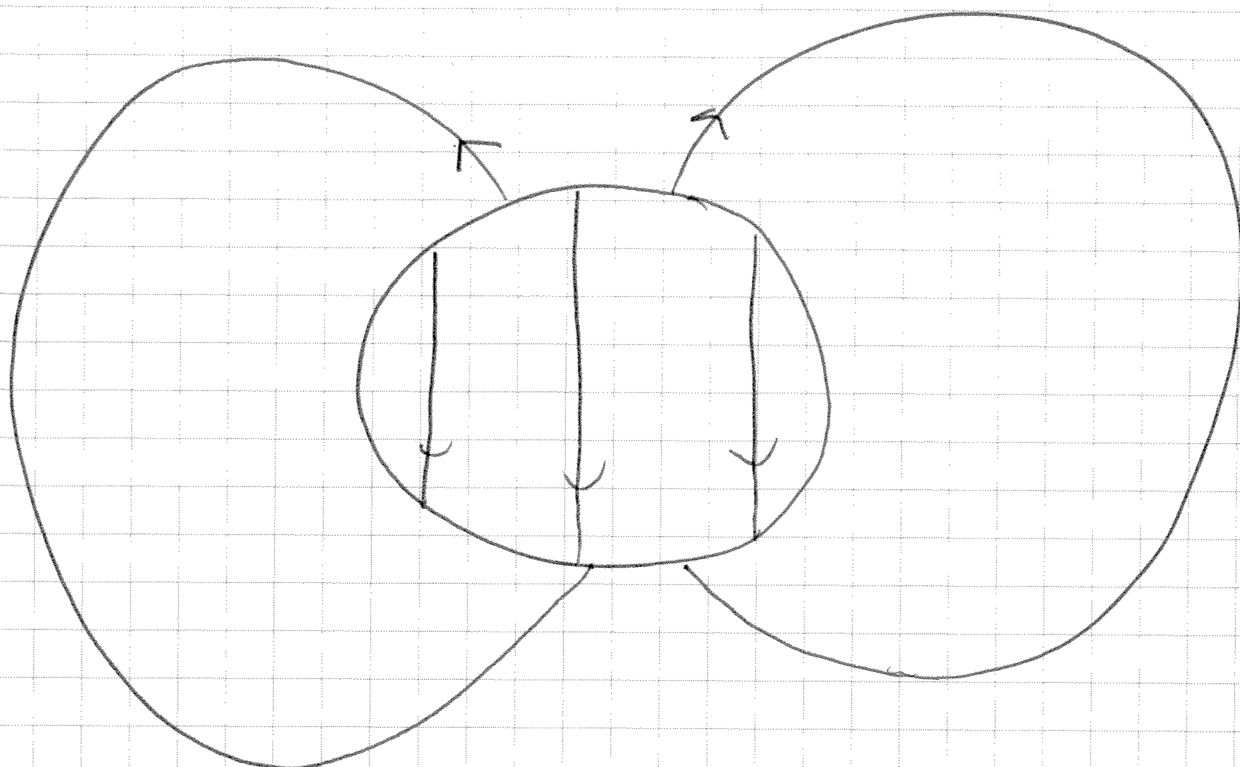
$$\vec{p} = \vec{P} V = \frac{4}{3} \pi R^3 P_0 \hat{z}$$

$$V_0(r, \theta) = \frac{|\vec{p}| \cos\theta}{4\pi\varepsilon_0 r^2}$$

$$\vec{p} \cdot \hat{r} = |\vec{p}| \cos\theta$$

$$V_0(r, \theta) = \frac{\vec{p} \cdot \hat{r}}{4\pi\varepsilon_0 r^2}$$

$\Longrightarrow$ The field outside is that of a pure dipole.



Now let's look at this more generally. For any object with polarization $\vec{P}(\vec{r}\,')$, the potential is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\vec{P}(\vec{r}\,') \cdot \hat{r}''}{(r'')^2}\, d\tau'$$

We can show

$$\nabla'\left(\frac{1}{r''}\right) = \frac{\hat{r}''}{(r'')^2}$$

Note, $\nabla' = \left(\frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right)$

So

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \vec{P}(\vec{r}') \cdot \nabla'\left(\frac{1}{r''}\right) d\tau'$$

<u>From front cover</u>

$$\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot \nabla f$$

So

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \nabla' \cdot \left(\frac{\vec{P}(\vec{r}')}{r''}\right) d\tau'$$

$$- \frac{1}{4\pi\varepsilon_0}\int_V \frac{1}{r''}\left(\nabla' \cdot \vec{P}(\vec{r}')\right) d\tau'$$

Apply Divergence Thm to First Term

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \oint_S \frac{1}{r''} \vec{P}(\vec{r}\,') \cdot d\vec{a}\,'$$

$$- \frac{1}{4\pi\epsilon_0} \int_V \frac{1}{r''} \nabla' \cdot \vec{P}(\vec{r}\,')\, d\tau'$$

- The first term looks like the potential of a surface charge

$$\sigma_b = \vec{P}(\vec{r}\,') \cdot \hat{n}$$

where $\hat{n}$ is the surface normal.

- The second term looks like the potential of a volume charge density

$$\rho_b = -\nabla' \cdot \vec{P}(\vec{r}\,')$$

These are real net charge densities called <u>bound</u> charge densities.

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \oint_S \frac{\sigma_b\, da'}{r''} + \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho_b\, d\tau'}{r''}$$

Check against our model systems

Uniform Polarized Slab

$$\sigma_b = \vec{P} \cdot \hat{z} = \rho d \quad \checkmark$$

Uniform Polarized Sphere

$$\sigma_b = \vec{P} \cdot \hat{r} = |\vec{P}| \cos\theta = \rho d \cos\theta$$

For both systems, $\rho_b = -\nabla \cdot \vec{P} = 0 \quad \checkmark$