Solution to E.1.1

$$\nabla\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right) \equiv -\vec{E} \qquad \vec{p} \text{ constant}$$

Three Possible Methods

I. All Cartesian

II. All Spherical

III Mixed

Method III

Product Rule $\quad \nabla(fg) = g\nabla f + f\nabla g$

Let $f = \frac{1}{r^3} \qquad g = \vec{p}\cdot\vec{r}$

$$-\vec{E} = (\vec{p}\cdot\vec{r})\,\nabla\left(\frac{1}{r^3}\right) + \left(\frac{1}{r^3}\right)\nabla(\vec{p}\cdot\vec{r})$$

In Cartesian, $\vec{p} = (p_x, p_y, p_z)$

$\vec{r} = (x, y, z)$

$$\vec{p}\cdot\vec{r} = xp_x + yp_y + zp_z$$

$$\nabla(\vec{p}\cdot\vec{r}) = \vec{p}$$

In Spherical

$$\nabla\left(\frac{1}{r^3}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r^3}\right)\hat{r} = -3\frac{1}{r^4}\hat{r}$$

$$-\vec{E} = (\vec{p}\cdot\vec{r})\left(\frac{-3}{r^4}\hat{r}\right) + \frac{1}{r^3}\cdot\vec{p}$$

$$= \frac{1}{r^3}\left(\vec{p} - 3(\vec{p}\cdot\hat{r})\hat{r}\right)$$

Method II All Cartesian

$$\vec{p} = (p_x, p_y, p_z) \qquad \vec{r} = (x, y, z)$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$V = \frac{p_x x + p_y y + p_z z}{(x^2+y^2+z^2)^{3/2}}$$

Take curl in cartesian.

Method III   All spherical

$$\vec{r} = r\,\hat{r} \qquad \vec{p} = p_r\,\hat{r} + p_\theta\,\hat{\theta} + p_\phi\,\hat{\phi}$$

$$\vec{p}\cdot\vec{r} = r\,p_r$$

$$V = \frac{r\,p_r}{r^3} = \frac{p_r}{r^2}$$

$\Rightarrow$ Problem: p_r may not be constant in spherical coordinates.

$$\vec{p} = p_x\,\hat{x} + p_y\,\hat{y} + p_z\,\hat{z} = p_r\,\hat{r} + p_\theta\,\hat{\theta} + p_\phi\,\hat{\phi}$$

$$p_r = \hat{r}\cdot\vec{p} = p_x(\hat{r}\cdot\hat{x}) + p_y(\hat{r}\cdot\hat{y}) + p_z(\hat{r}\cdot\hat{z})$$

From Griffiths Transformation Equations

$$\hat{x}\cdot\hat{r} = \sin\theta\cos\phi$$

$$\hat{y}\cdot\hat{r} = \sin\theta\sin\phi$$

$$\hat{z}\cdot\hat{r} = \cos\theta$$

$$p_r = p_x\sin\theta\cos\phi + p_y\sin\theta\sin\phi + p_z\cos\theta$$

$$V = \frac{P_x \sin\theta\cos\phi + P_y \sin\theta\sin\phi + P_z \cos\theta}{r^2}$$

Take Gradient in Spherical

Maple solutions to Cartesian and Spherical gradients follow.

Cartesian

```
> assume(px, real);
> assume(py, real);
> assume(pz, real);
> with(VectorCalculus);
Warning, the assigned names `<,>` and `<|>` now have a global binding
Warning, these protected names have been redefined and unprotected: `*`, `+`, `-`, `.`, D, Vector, diff, int, limit, series
```

[&x, *, +, -, ., <,>, <|>, AddCoordinates, ArcLength, BasisFormat, Binormal, CrossProd, CrossProduct, Curl, Curvature, D, Del, DirectionalDiff, Divergence, DotProd, DotProduct, Flux, GetCoordinateParameters, GetCoordinates, Gradient, Hessian, Jacobian, Laplacian, LineInt, MapToBasis, Nabla, Norm, Normalize, PathInt, PrincipalNormal, RadiusOfCurvature, ScalarPotential, SetCoordinateParameters, SetCoordinates, SurfaceInt, TNBFrame, Tangent, TangentLine, TangentPlane, TangentVector, Torsion, Vector, VectorField, VectorPotential, Wronskian, diff, evalVF, int, limit, series] (1)

> $p := < px,\ py,\ px >;$

$$p := (px\sim)e_x + (py\sim)e_y + (px\sim)e_z \tag{2}$$

> $r := < x, y, z >;$

$$r := (x)e_x + (y)e_y + (z)e_z \tag{3}$$

> $p.r;$

$$px\sim x + py\sim y + px\sim z \tag{4}$$

> $V := \dfrac{p.r}{Norm(r)^3};$

$$V := \frac{px\sim x + py\sim y + px\sim z}{\left(x^2+y^2+z^2\right)^{(3/2)}} \tag{5}$$

> $SetCoordinates(cartesian[x, y, z]);$

$$cartesian_{x,\,y,\,z} \tag{6}$$

> $Gradient(V);$

$$\left(\frac{px\sim}{\left(x^2+y^2+z^2\right)^{(3/2)}} - \frac{3\ (px\sim x + py\sim y + px\sim z)\,x}{\left(x^2+y^2+z^2\right)^{(5/2)}}\right)\bar{e}_x + \left(\frac{py\sim}{\left(x^2+y^2+z^2\right)^{(3/2)}} - \frac{3\ (px\sim x + py\sim y + px\sim z)\,y}{\left(x^2+y^2+z^2\right)^{(5/2)}}\right)\bar{e}_y + \left(\frac{px\sim}{\left(x^2+y^2+z^2\right)^{(3/2)}} - \frac{3\ (px\sim x + py\sim y + px\sim z)\,z}{\left(x^2+y^2+z^2\right)^{(5/2)}}\right)\bar{e}_z \tag{7}$$

> $simplify(\%);$

$$\left(\frac{-2\,px\sim x^2 + px\sim y^2 + px\sim z^2 - 3\,x\,py\sim y - 3\,z\,px\sim x}{\left(x^2+y^2+z^2\right)^{(5/2)}}\right)\bar{e}_x + \left(\frac{py\sim x^2 - 2\,py\sim y^2 + py\sim z^2 - 3\,y\,px\sim x - 3\,y\,px\sim z}{\left(x^2+y^2+z^2\right)^{(5/2)}}\right)\bar{e}_y \tag{8}$$

$$-\frac{-px{\sim}\,x^2 - px{\sim}\,y^2 + 2\,px{\sim}\,z^2 + 3\,z\,px{\sim}\,x + 3\,z\,py{\sim}\,y}{\left(x^2+y^2+z^2\right)^{(5/2)}}\,\bar{e}_z$$

>

Note, px~ means px is a variable about which some assumptions have been made.

> *with*(*VectorCalculus*);

```
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binding
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```

[&x, *, +, -, ., <,>, <|>, *AddCoordinates, ArcLength, BasisFormat, Binormal, CrossProd, CrossProduct, Curl, Curvature*, D, *Del, DirectionalDiff, Divergence, DotProd, DotProduct, Flux, GetCoordinateParameters, GetCoordinates, Gradient, Hessian, Jacobian, Laplacian, LineInt, MapToBasis, Nabla, Norm, Normalize, PathInt, PrincipalNormal, RadiusOfCurvature, ScalarPotential, SetCoordinateParameters, SetCoordinates, SurfaceInt, TNBFrame, Tangent, TangentLine, TangentPlane, TangentVector, Torsion, Vector, VectorField, VectorPotential, Wronskian, diff, evalVF, int, limit, series*] (1)

> *SetCoordinates*(*'spherical'*$[r, \phi, \theta]$);

$$spherical_{r,\,\phi,\,\theta} \tag{2}$$

> $V := \dfrac{(px \cdot sin(theta) \cdot cos(phi) + py \cdot sin(theta) \cdot sin(phi) + pz \cdot cos(theta))}{r^2};$

$$V := \frac{px \sin(\theta)\cos(\phi) + py \sin(\theta)\sin(\phi) + pz\cos(\theta)}{r^2} \tag{3}$$

> *Gradient*(*V*);

$$-\frac{2\,(px\sin(\theta)\cos(\phi) + py\sin(\theta)\sin(\phi) + pz\cos(\theta))}{r^3}\,\bar{e}_r + \left(\frac{-px\sin(\theta)\sin(\phi) + py\sin(\theta)\cos(\phi)}{r^3}\right)\bar{e}_\phi + \left(\frac{px\cos(\theta)\cos(\phi) + py\cos(\theta)\sin(\phi) - pz\sin(\theta)}{r^3\sin(\phi)}\right)\bar{e}_\theta \tag{4}$$

>