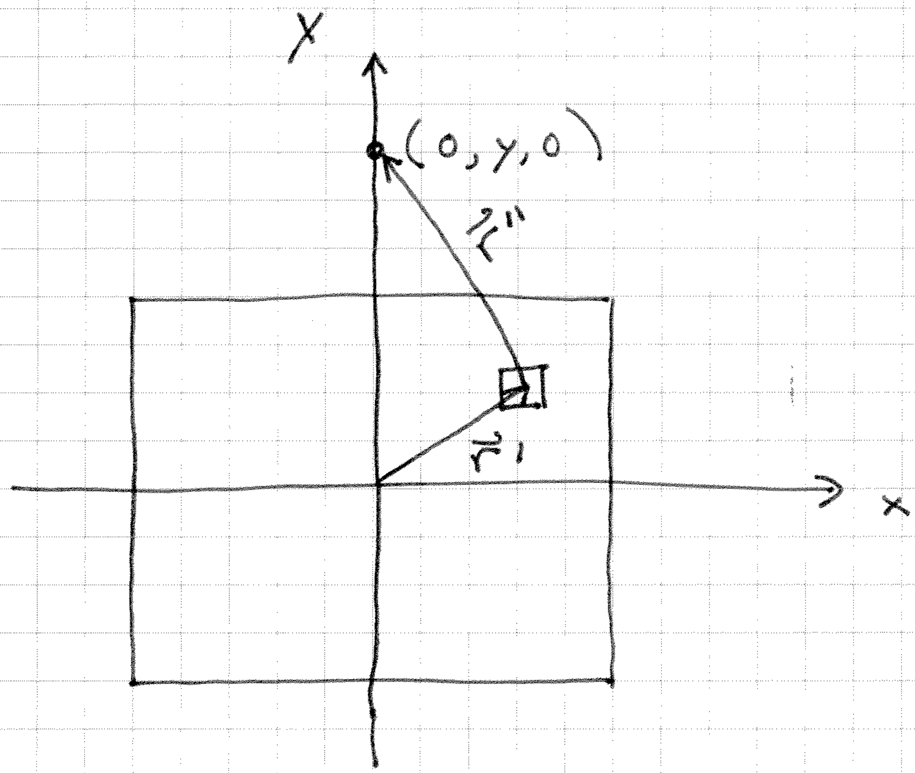
Solution to E.2.3

A square with uniform surface charge density σ with $x \in [-a, a]$, $y \in [-a, a]$. Compute field at points along y axis with $y > 0$.



Field Point $\quad \vec{r} = (0, y, 0)$

Source Point $\quad \vec{r}\,' = (x', y', 0)$

Displacement $\quad \vec{r}\,'' = \vec{r} - \vec{r}\,' = (-x', y - y', 0)$

Length $\quad r'' = \sqrt{(x')^2 + (y - y')^2}$

Coulomb's Law

$$\vec{E}(\vec{r}) = \int \frac{\sigma\, da'\, \hat{r}''}{4\pi\varepsilon_0 r''^2}$$

$$= \frac{\sigma}{4\pi\varepsilon_0} \int \frac{da'\, \vec{r}''}{(r'')^3}$$

$$= \frac{\sigma}{4\pi\varepsilon_0} \int_{-a}^{a} dx' \int_{-a}^{a} dy' \frac{1}{(x'^2 + (y-y')^2)^{3/2}} (-x', y-y', 0)$$

$$= \frac{\sigma \hat{x}}{4\pi\varepsilon_0} \int_{-a}^{a} dx' \int_{-a}^{a} dy' \frac{-x'}{(x'^2 + (y-y')^2)^{3/2}}$$

$$+ \frac{\sigma \hat{y}}{4\pi\varepsilon_0} \int_{-a}^{a} dx' \int_{-a}^{a} dy' \frac{y-y'}{(x'^2 + (y-y')^2)^{3/2}}$$

$\hat{x}$ integral is zero

I. By symmetry field points in $+\hat{y}$ direction

II. Integral of odd function over even range $= 0$

$\Rightarrow$ Nice when the math and physics check.

$$\vec{E} = \frac{\sigma \hat{y}}{4\pi\varepsilon_0} \int_{-a}^{a} dx' \int_{-a}^{a} dy' \frac{y-y'}{\left(x'^2 + (y-y')^2\right)^{3/2}}$$

<u>u-sub</u> $\quad u = y - y' \qquad du = -dy'$

$$\vec{E} = -\frac{\sigma \hat{y}}{4\pi\varepsilon_0} \int_{-a}^{a} dx' \int_{y+a}^{y-a} du \frac{u}{\left(x'^2 + u^2\right)^{3/2}}$$

$$= \frac{-\sigma \hat{y}}{4\pi\varepsilon_0} \int_{-a}^{a} dx' \left[-\frac{1}{\sqrt{x'^2 + u^2}} \right] \Bigg|_{y+a}^{y-a}$$

$$= \frac{\sigma \hat{y}}{4\pi\varepsilon_0} \int_{-a}^{a} dx' \left[\frac{1}{\sqrt{x'^2 + (y-a)^2}} - \frac{1}{\sqrt{x'^2 + (y+a)^2}} \right]$$

$\Rightarrow$ Even integral ($x' \to -x'$ Leaves unchanged)

$$\vec{E} = \frac{\sigma \hat{y}}{2\pi\varepsilon_0} \int_{0}^{a} dx' \left[\frac{1}{\sqrt{x'^2 + (y-a)^2}} - \frac{1}{\sqrt{x'^2 + (y+a)^2}} \right]$$

$$\int \frac{dx}{\sqrt{x^2 + R^2}} = \ln\left(x + \sqrt{x^2 + R^2} \right) + C$$

$$\vec{E} = \frac{\sigma \hat{y}}{2\pi\varepsilon_0} \left[\ln\left(x' + \sqrt{x'^2 + (y-a)^2}\right) - \ln\left(x' + \sqrt{x'^2 + (y+a)^2}\right] \right|_0^a$$

$$= \frac{\sigma \hat{y}}{2\pi\varepsilon_0} \left[\ln\left(a + \sqrt{a^2 + (y-a)^2} - \ln(|y-a|) - \ln\left(a + \sqrt{a^2 + (y+a)^2} + \ln(|y+a|)\right) \right.$$

$|y-a| = y-a$ since $y > a$.

$$\vec{E} = \frac{\sigma \hat{y}}{2\pi\varepsilon_0} \left[\ln\left(\frac{a + \sqrt{a^2+(y-a)^2}}{y-a}\right) - \ln\left(\frac{a + \sqrt{a^2+(y+a)^2}}{y+a}\right) \right]$$

Checks

- Units

$[\ ]$ dimensionless

$$\frac{\sigma}{\varepsilon_0} \quad \frac{C/m^2}{C^2/Nm^2} = \frac{N}{C} \checkmark$$

• Limit $y \to \infty$ - Collect small parameter

$$\vec{E} = \frac{\sigma \hat{y}}{2\pi\varepsilon_0}\left[\ln\left(\frac{a}{y-a} + \sqrt{1+\left(\frac{a}{y-a}\right)^2}\right.\right.$$

$$\left.\left. - \ln\left(\frac{a}{y+a} + \sqrt{1+\left(\frac{a}{y+a}\right)^2}\right)\right]$$

<u>Binomial Expansion</u>

$$(1+x)^n \sim 1 + nx + \cdots$$

$$\vec{E} = \frac{\sigma \hat{y}}{2\pi\varepsilon_0}\left(\ln\left(\frac{a}{y-a} + 1 + \frac{1}{2}\left(\frac{a}{y-a}\right)^2 + \cdots\right)\right.$$

$$\left. - \ln\left(\frac{a}{y+a} + 1 + \frac{1}{2}\left(\frac{a}{y+a}\right)^2 + \cdots\right)\right)$$

<u>Expand log</u>

$$\ln(1+x) = x - \frac{x^2}{2} + \cdots$$

Try keeping $O(1)$ in small parameter

$$\vec{E} = \frac{\sigma \hat{y}}{2\pi\varepsilon_0}\left[\frac{a}{y-a} - \frac{a}{y+a} + O(2)\right]$$

$$= \frac{\sigma \hat{y}}{2\pi\varepsilon_0}\left[\frac{a(y+a) - a(y-a)}{y^2-a^2}\right]$$

$$= \frac{\sigma \hat{y}}{2\pi\varepsilon_0}\left(\frac{2a^2}{y^2-a^2}\right) \sim \frac{\sigma \hat{y}}{2\pi\varepsilon_0}\left(\frac{2a^2}{y^2}\right)$$

$$= \frac{4\sigma a^2}{4\pi\varepsilon_0 y^2}\hat{y} = \frac{Q}{4\pi\varepsilon_0 y^2}\hat{y} \quad \checkmark$$

Field of a point charge

```
(%i1) assume(a>0);
      assume(y>a);
      integrate((y-yp)/(xp^2+(y-yp)^2)^(3/2), yp, -a, a);
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(%o1) $[a>0]$

(%o2) $[y>a]$

Is xp zero or nonzero? nonzero;

(%o3) $$\frac{\sqrt{y^2+2\,a\,y+xp^2+a^2}-\sqrt{y^2-2\,a\,y+xp^2+a^2}}{\sqrt{y^2-2\,a\,y+xp^2+a^2}\,\sqrt{y^2+2\,a\,y+xp^2+a^2}}$$

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(%i4) integrate(%o3, xp, -a, a);
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(%o4) $$2\,\mathrm{asinh}\left(\frac{a}{\sqrt{y^2-2\,a\,y+a^2}}\right)-2\,\mathrm{asinh}\left(\frac{a}{\sqrt{y^2+2\,a\,y+a^2}}\right)$$