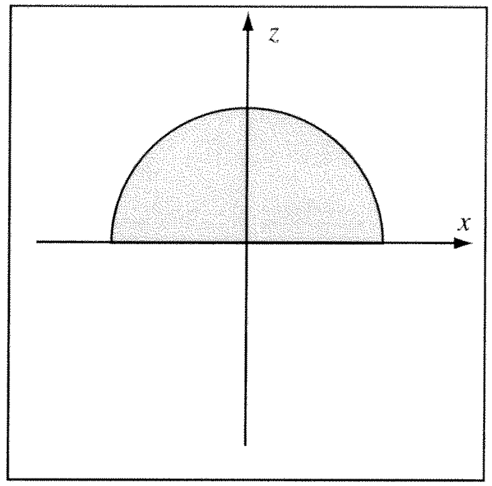
Electricity and Magnetism - Test 1 - Spring 2014

Work four of the six problems. Place the problems in the order you wish them graded. The first two problems form the first half test; the second two problems form the second half test.

Problem 1.1 Compute the electric potential at the origin of an non-uniformly charged half-sphere with charge density $\rho(r)$ where $\rho(r)$ is non-zero at points $r < a$ and $z > 0$. ρ is given by

$$\rho = \frac{\gamma \sin(\theta)}{r}$$

and γ is a constant.



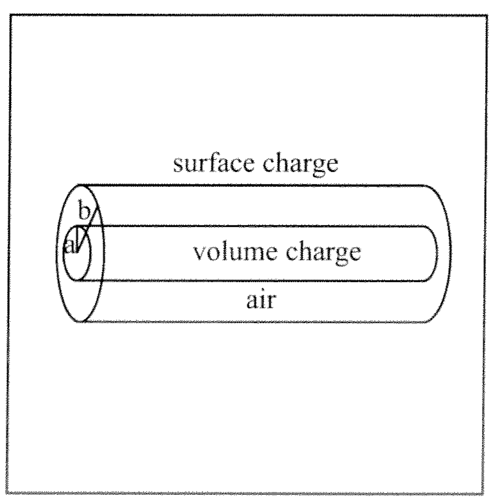
Problem 1.2 Consider the electric field $\vec{E} = \gamma \cos^2(\theta)\hat{r}$ given in spherical coordinates. Is this a possible electrostatic field? If not, why? If it is, find the charge density that produced the field.

Problem 1.3 Consider the non-uniform volume change $\rho(z) = \gamma \sin(kz)$ for $-a < z < a$ where γ is a constant and $k = \pi/a$. $\rho = 0$ for $z < -a$ and $z > a$. The charge density is constant in the x and y directions. Compute the electric field everywhere.

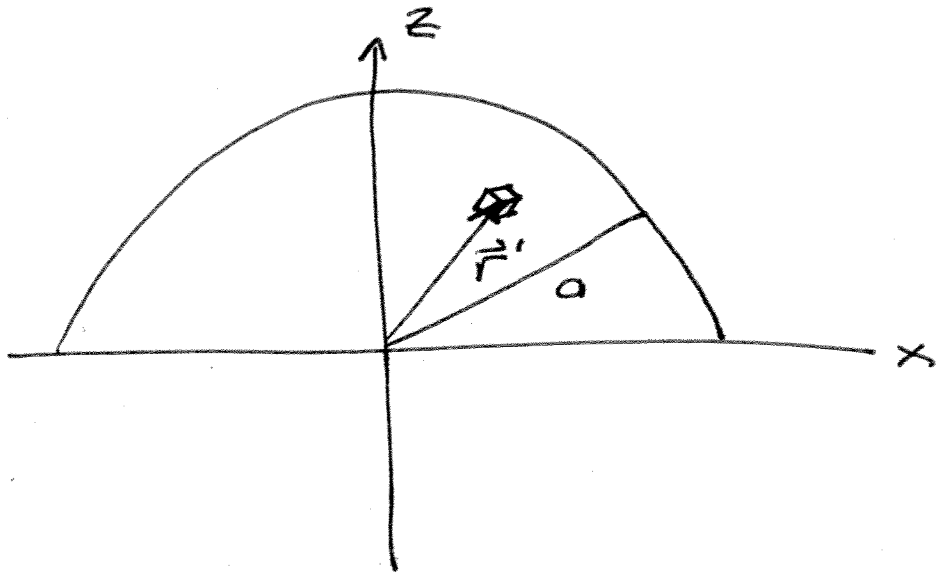
Problem 1.4 A non-uniformly charged disk lies in the $x - y$ plane centered at the origin. The disk has charge density $\sigma = \gamma/s$ in cylindrical coordinates and is of radius a. Compute the electric field a distance R along the positive z axis.

Problem 1.5 Calculate the energy stored in the electric fields INSIDE a uniform spherical volume charge of radius a and charge density ρ.

Problem 1.6 A cylindrically symmetric system has a uniform volume charge density ρ for $s < a$ where a is the radius of the cylinder. It has $\rho = 0$ for $a < s < b$ and $s > b$. It has a cylinder concentric with the volume charge density of radius b and uniform surface charge density σ. Compute the electric field everywhere.



(1)



$$\rho = \frac{\gamma \sin\theta}{r}$$

<u>Source Point</u> $\quad \vec{r}' = r'\hat{r}'$

<u>Field Point</u> $\quad \vec{r} = 0$

<u>Displacement</u> $\quad \vec{r}'' = 0 - \vec{r}' = -\vec{r}'$

<u>Length</u> $\quad r'' = r'$

<u>Area Element</u> $\quad d\tau' = (dr')(r'd\theta')(r'\sin\theta' d\phi')$

<u>Potential</u>

$$V(0) = \int \frac{\rho \, d\tau'}{4\pi\epsilon_0 r''}$$

$$= \frac{1}{4\pi\epsilon_0} \int_0^a dr' \int_0^{2\pi} d\phi' \int_0^{\pi/2} d\theta' \left(r'^2 \sin\theta'\right)\left(\frac{\gamma \sin\theta'}{r'}\right)\frac{1}{r'}$$

$$V(0) = \frac{\gamma}{4\pi\epsilon_0} \underbrace{\int_0^a dr'}_{a} \; \underbrace{\int_0^{2\pi} d\phi'}_{2\pi} \; \underbrace{\int_0^{\pi/2} d\theta' \sin^2\theta'}_{\pi/4}$$

Math handbook

$$= \frac{\gamma \pi a}{8\epsilon_0}$$

(2) $\vec{E} = \gamma \cos^2\theta \, \hat{r}$

$$\nabla \times \vec{E} = \frac{1}{r\sin\theta} \frac{\partial v_r}{\partial \phi} \hat{\theta}$$

$$- \frac{1}{r} \frac{\partial v_r}{\partial \theta} \hat{\phi}$$

$v_r = \gamma \cos^2\theta$

$v_\theta = 0$

$v_\phi = 0$

$$= \frac{2\gamma}{r} \cos\theta \sin\theta \, \hat{\phi}$$

since $\nabla \times \vec{E} \neq 0$, not a possible electrostatic field

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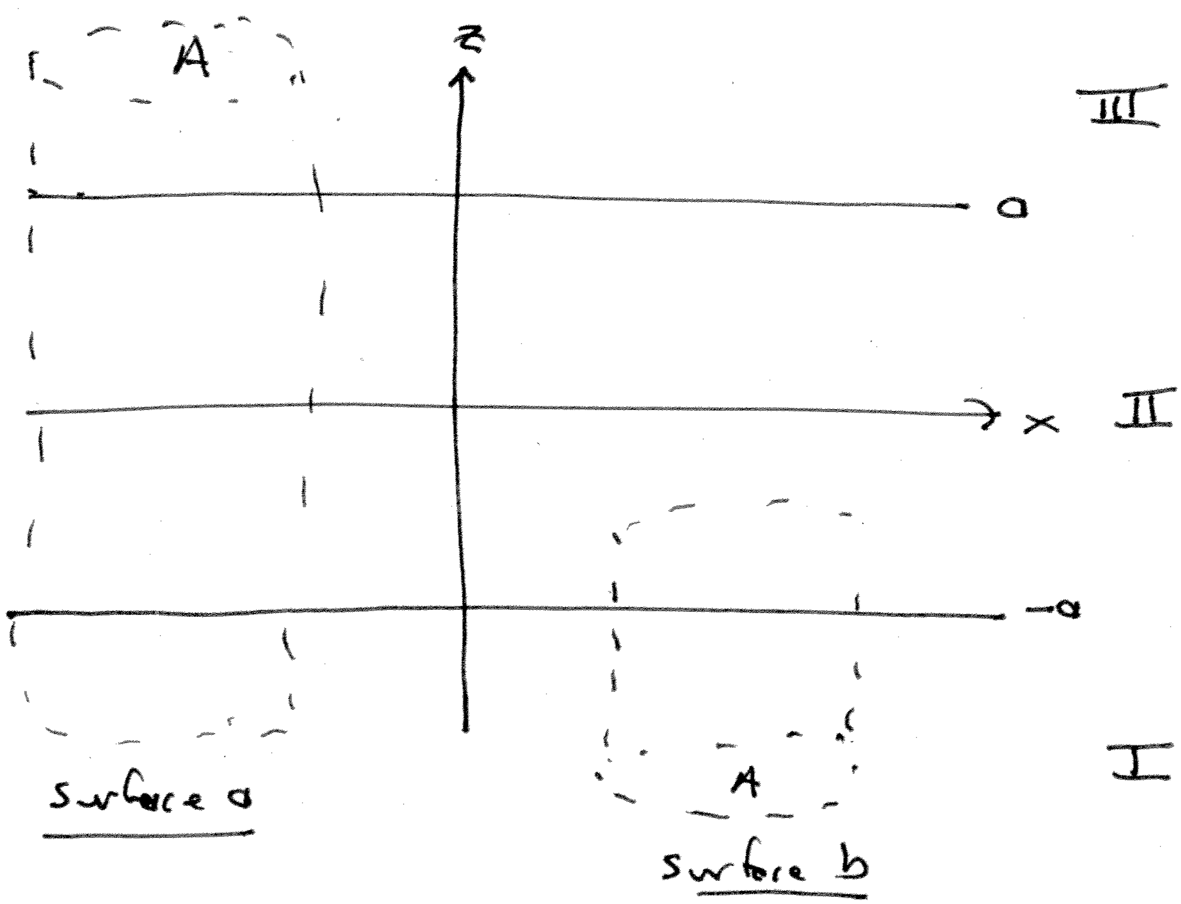
(3) Consider the non-uniform volume charge density $\rho(z) = \gamma \sin kz$ for $-a < z < a$ and $\rho = 0$ for $z < -a$ and $z > a$. γ is a constant and $k = \pi/a$. Compute the electric field everywhere.



Use Gauss' Law. The charge enclosed in surface a is

$$Q_{enc} = \int \rho \, d\tau = A \int_{-a}^{a} \rho \, dz$$

$$Q_{enc} = A\gamma \int_{-a}^{a} \sin kz \, dz = 0$$

<u>Gauss' Law</u> $\Phi = E_{III} A - E_{I} A = 0$

Symmetry $E_{III} = -E_{I}$

$$\Rightarrow \quad \vec{E}_{I} = 0 \qquad \vec{E}_{III} = 0$$

<u>Surface b</u> Let the upper edge of surface b be at z.

$$Q_{enc} = A \int_{-a}^{z} \rho(z') \, dz'$$

$$= A\gamma \int_{-a}^{z} \sin kz' \, dz'$$

$$= -\frac{A\gamma}{k} \cos kz' \Big|_{-a}^{z}$$

$$= -\frac{A\gamma}{k} \left(\cos kz - \cos(-ak) \right)$$

$$= -\frac{A\gamma}{k} \left(\cos kz - \cos(-\pi) \right)$$

$$= -\frac{A\gamma}{k} \left(\cos kz + 1 \right)$$

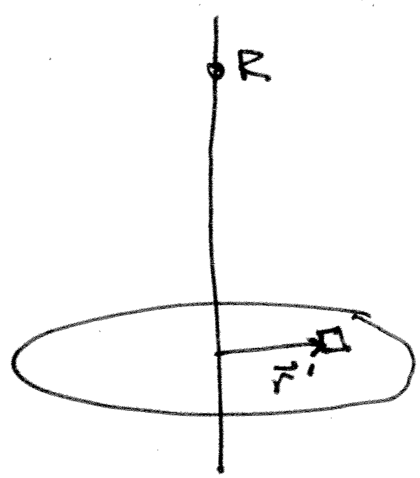
Gauss

$$\Phi = E_{II}A - E_{I}A = E_{II}A = \frac{Q_{enc}}{\varepsilon_0}$$

$$= -\frac{A\gamma}{\varepsilon_0 k}\left(\cos kz + 1\right)$$

$$\vec{E}_{II} = \frac{-\gamma}{\varepsilon_0 k}\left(\cos kz + 1\right)\hat{z}$$

(4) A disk of radius a concentric with the z axis and lying in the x-y plane has non-uniform surface charge density $\sigma = \gamma / s$. Compute the electric field a distance R along the positive z axis.



Field Point $\quad \vec{r} = R\hat{z}$

Source Point $\quad \vec{r}\,' = s'\hat{s}'$

Displacement $\quad \vec{r}\,'' = \vec{r} - \vec{r}\,' = R\hat{z} - s'\hat{s}'$

Length $\quad r'' = \sqrt{R^2 + s'^2}$ $\quad$ since $\hat{z} \perp \hat{s}'$

Coulomb

$$\vec{E} = \int \frac{\sigma\, da'}{4\pi\varepsilon_0 (r'')^3}\, \vec{r}\,''$$

$$da' = ds'\, s' d\phi'$$

$$\vec{E} = \int_0^a ds' \int_0^{2\pi} d\phi' \; \frac{s' \cdot \gamma / s'^{2} \quad \vec{r}''}{4\pi\epsilon_0 \left(\sqrt{s'^2 + R^2}\right)^3}$$

The field must point in the z direction by symmetry, so the $\hat{s}'$ integral must be zero.

$$\vec{E} = \frac{\hat{z}\, \gamma R}{4\pi\epsilon_0} \left[\int_0^a ds' \frac{1}{(s'^2 + R^2)^{3/2}} \right] \left[\int_0^{2\pi} d\phi' \right]$$

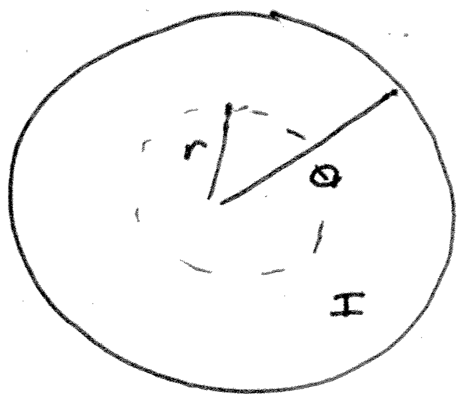
$$\qquad\qquad = \qquad\qquad\qquad =$$

$$\frac{x}{R^2 \sqrt{x^2 + R^2}} \Bigg|_0^a \qquad 2\pi$$

$$\vec{E} = \frac{2\pi\gamma}{4\pi\epsilon_0 R} \frac{a}{\sqrt{a^2 + R^2}} \hat{z}$$

$$= \frac{\gamma}{2\epsilon_0 R} \frac{a}{\sqrt{a^2 + R^2}} \hat{z}$$

(5) A uniform volume charge ρ has radius a. Compute the energy stored in the fields inside the volume charge.



Fields $\quad Q_{enc} = \frac{4}{3}\pi r^3 \rho$

Gauss $\quad \vec{E}_I(r<a) = \dfrac{\frac{4}{3}\pi r^3 \rho}{4\pi \epsilon_0 r^2}\hat{r}$

$$= \frac{\rho r}{3\epsilon_0}\hat{r}$$

Energy Density

$$u = \frac{1}{2}\epsilon_0 E^2 = \frac{1}{2}\epsilon_0 \frac{\rho^2 r^2}{9\epsilon_0^2}$$

$$= \frac{1}{18\epsilon_0}\rho^2 r^2$$

Integrate energy density

$$U = \int_{\text{sphre}} u \, d\tau = \int_0^a dr \, 4\pi r^2 u$$

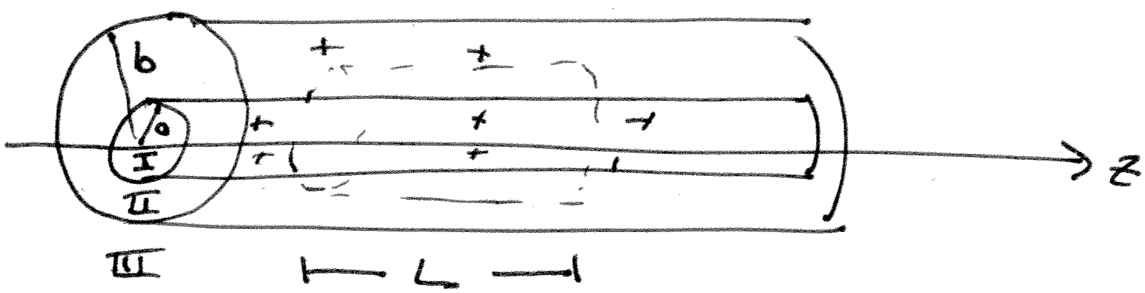
spherically symmetric

$$= 4\pi \left(\frac{\rho^2}{18 \varepsilon_0} \right) \int_0^a r^4 dr =$$

$$U = \frac{2}{9} \pi \frac{\rho^2}{\varepsilon_0} \frac{a^5}{5} = \frac{2}{45} \pi \frac{\rho^2 a^5}{\varepsilon_0}$$

(6) A cylindrical system has uniform volume charge density ρ for $s<a$, $\rho=0$ for $a<s<b$ and $s>b$. The system has a cylindrical uniform surface charge density σ at a radius b.

Compute the field everywhere.



Use a cylindrical Gaussian surface of radius s and length L.

Region I ($s<a$) $Q_{enc} = \pi s^2 L \rho$

$$\Phi = 2\pi s L E_I = \frac{Q_{enc}}{\varepsilon_0} = \frac{\pi s^2 L \rho}{\varepsilon_0} \quad \text{Gauss}$$

$$\vec{E}_I = \frac{\rho s}{2\varepsilon_0}\hat{s}$$

Region II $(a<s<b)$ $Q_{enc} = \pi a^2 \rho L$

$$\Phi = 2\pi s L E_{II} = \frac{Q_{enc}}{\epsilon_0} = \frac{\pi a^2 \rho L}{\epsilon_0}$$

$$\vec{E}_{II} = \frac{\rho a^2}{2\epsilon_0 s}\hat{s}$$

Region III $s>b$

$$Q_{enc} = \pi a^2 \rho L + 2\pi b L \sigma$$

Gauss $E_{III} = \dfrac{Q_{enc}}{2\pi s \epsilon_0 L}$

$$\vec{E}_{III} = \frac{\pi a^2 \rho + 2\pi b \sigma}{2\pi \epsilon_0 s}\hat{s}$$