Vector Potential

Since $\nabla \cdot \vec{B} = 0$, the magnetic field is uniquely determined by the curl of some vector function $\vec{A}$ called the vector potential.

$$\vec{B} = \nabla \times \vec{A}$$

Try

$$\vec{A} = \int_V \frac{\mu_0}{4\pi} \frac{\vec{J}(\vec{r}\,')}{r''} \, d\tau'$$

$$\nabla \times \vec{A} = \frac{\mu_0}{4\pi} \int_V \nabla \times \left(\frac{\vec{J}(\vec{r}\,')}{r''} \right) d\tau'$$

Vector Identity

$$\nabla \times (f\vec{A}) = f \, \nabla \times \vec{A} - \vec{A} \times \nabla f$$

$$\nabla \times \vec{A} = \frac{\mu_0}{4\pi} \int_V \left[\frac{1}{r''} \nabla \times \vec{J}(\vec{r}\,') - \vec{J}(\vec{r}\,') \, \nabla\left(\frac{1}{r''}\right) \right] d\tau'$$

The first term is zero because the curl is with respect to unprimed coordinates but $\vec{J}(\vec{r}')$ depends on primed coordinates.

As before, $\nabla\left(\frac{1}{r''}\right) = -\frac{\hat{r}''}{r''^2}$

$$\nabla \times \vec{A} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}(\vec{r}') \times \hat{r}''}{(r'')^2} \, d\tau' = \vec{B}$$

We can also write

$$\vec{A} = \frac{\mu_0}{4\pi} \int_C \frac{I \, d\vec{l}'}{r''} = \frac{\mu_0}{4\pi} \int_C \frac{\vec{I} \, dl'}{r''}$$

$$\vec{A} = \frac{\mu_0}{4\pi} \int_S \frac{\vec{K} \, da'}{r''}$$

Qualitatively, ~~B points~~ if you point your thumb in the direction of $\vec{B}$, your fingers curl in the direction of $\vec{A}$.

Integral Form Vector Potential

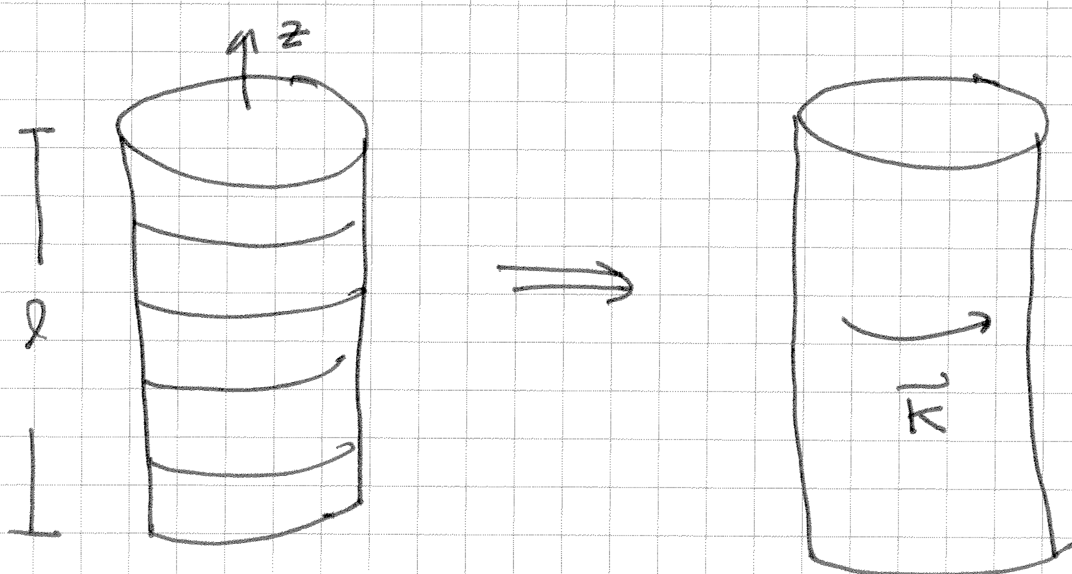
$$\int_S (\nabla \times \vec{A}) \cdot d\vec{a} = \int_S \vec{B} \cdot d\vec{a} = \Phi_m$$

$$\parallel \text{ Stokes}$$

$$\oint_C \vec{A} \cdot d\vec{l}$$

$\Rightarrow$ We can use reasoning similar to Ampere's Law to find $\vec{A}$

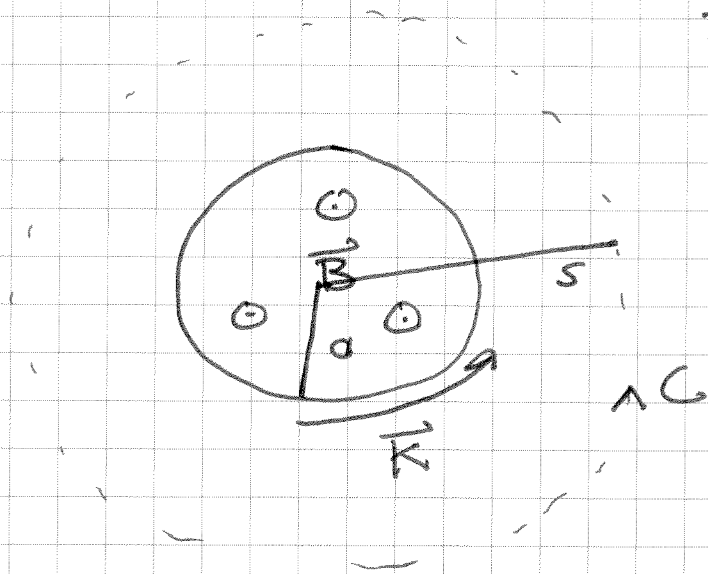
<u>Ex</u> Vector potential of infinite solenoid carrying current I with N turns per length l.



N turns

$$\vec{K} = \frac{N}{l} I \hat{\phi} = n I \hat{\phi}$$

End View



Positive normal out of page
$\hat{n} = \hat{z}$

By symmetry, $\vec{A}$ must circle $\vec{B} \Rightarrow \vec{A}$ circular.

Outside $(s > a)$

$$\oint_C \vec{A} \cdot d\vec{l} = 2\pi s A = \Phi_m = BA$$

$$= B\pi a^2$$

$$A_i = \frac{\pi a^2 B}{2\pi s} = \frac{1}{2} \frac{Ba^2}{s} = \frac{\frac{1}{2} \mu_0 K a^2}{s}$$

$$\vec{A}_i = \frac{\mu_0 K a^2}{2s} \hat{\phi}$$

Inside Solenoid $(s<a)$

$$\oint_c \vec{A}\cdot d\vec{l} = 2\pi s A = \Phi_m = B\pi s^2$$

$$\vec{A}_0 = \frac{sB}{2}\hat{\phi} = \frac{\mu_0 K s}{2}\hat{\phi}$$

Check Field

Inside

$$\nabla\times\vec{A}_i = \frac{1}{s}\left(\frac{\partial(sA_\phi)}{\partial s}\right)\hat{z}$$

$$= \frac{1}{s}\left(\frac{\partial}{\partial s}\frac{s^2 B}{2}\right)\hat{z}$$

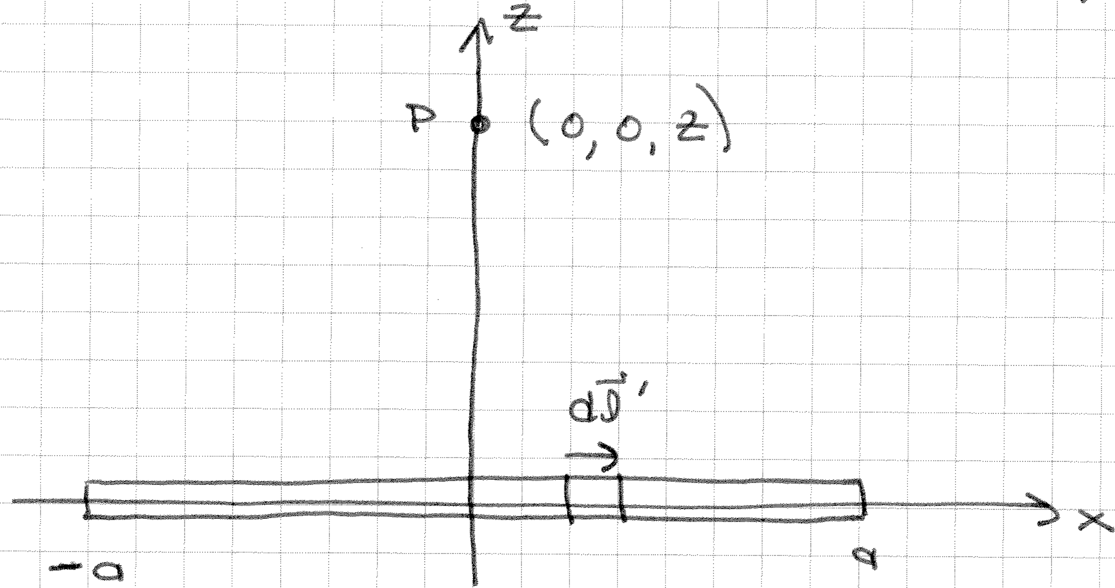
$$= B\hat{z} \checkmark$$

Outside

$$\nabla\times\vec{A}_o = \frac{1}{s}\left(\frac{\partial}{\partial s}\left(s\cdot\frac{1}{2}\frac{Ba^2}{s}\right)\right)\hat{z}$$

$$= 0 \checkmark$$

Ex Vector Potential of Finite Wire Along Axis



$$\vec{A} = \frac{\mu_0 I}{4\pi} \int_{-a}^{a} \frac{d\vec{l}'}{r''}$$

$$d\vec{l}' = dx' \, \hat{x}$$

$$\vec{r}' = (x', 0, 0) \qquad \vec{r} = (0, 0, z)$$

$$\vec{r}'' = (-x', 0, z)$$

$$r'' = \sqrt{(x')^2 + z^2}$$

$$\vec{A} = \frac{\mu_0 I}{4\pi} \hat{x} \int_{-a}^{a} \frac{dx'}{\sqrt{(x')^2 + z^2}}$$

An integral we've encountered before

$$\vec{A} = \frac{\mu_0 I}{2\pi} \ln\left[\frac{a + \sqrt{a^2+z^2}}{z}\right]\hat{x}$$

If the wire segment becomes long, $a \to \infty$, then

$$\sqrt{a^2+z^2} \longrightarrow a$$

and $$\vec{A} \longrightarrow \frac{\mu_0 I}{2\pi} \ln\left(\frac{2a}{z}\right)\hat{x}$$

$$= -\frac{\mu_0 I}{2\pi} \ln(z)\,\hat{x} + \text{constant}$$

Magnetic Field (cylindrical)

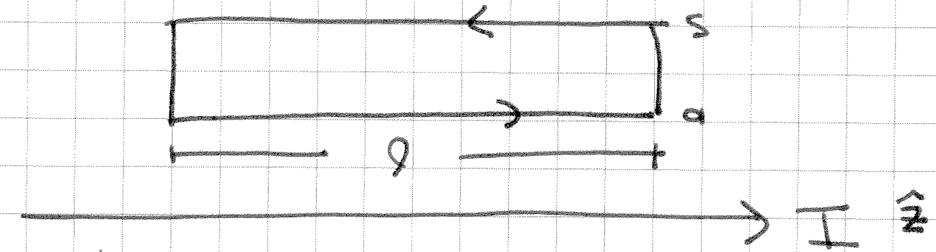
$$\vec{B} = \nabla \times \vec{A} = \frac{1}{s}\frac{\partial A_z}{\partial \phi}\hat{s} - \frac{\partial A_z}{\partial s}\hat{\phi}$$

if $$\vec{A} = -\frac{\mu_0 I}{2\pi}\ln(s)\,\hat{z}$$

exchanging $\hat{x} \longleftrightarrow \hat{z}$

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

<u>Ex</u> Let's try the infinite wire using the magnetic flux.



$$\Phi_m = l \int_a^s B(s)\, ds = \frac{\mu_0 I l}{2\pi} \int_a^s \frac{ds}{s}$$

$$= \frac{l \mu_0 I}{2\pi} \ln\left(\frac{s}{a}\right)$$

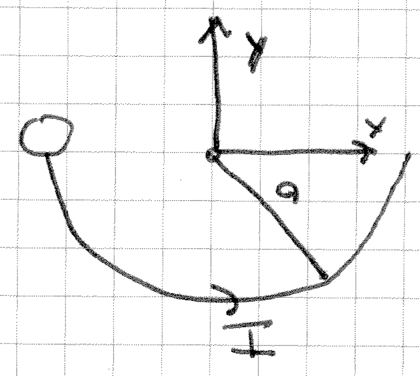
$$= \oint \vec{A} \cdot d\vec{l} = A(a)\, l - A(s)\, l$$

$$A(s) = \left(A(a)\, \hat{z} - \frac{\mu_0 I}{2\pi} \ln\left(\frac{s}{a}\right) \hat{z} \right)$$

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{s} \frac{\partial A_z}{\partial \phi} \hat{s} - \frac{\partial A_z}{\partial s} \hat{\phi}$$

$$= \frac{\mu_0 I}{2\pi s} \hat{\phi}$$

Ex Compute Vector Potential at center of circular segment of radius a carrying current I.



$$\vec{A} = \frac{\mu_0}{4\pi}\int_C \frac{I\, d\vec{l}\,'}{r''}$$

$r'' = a$

$d\vec{l}\,' = a\, d\phi'\, \hat{\phi}'$

$$= \frac{\mu_0 I}{4\pi}\int_{\pi}^{2\pi} \frac{a\, d\phi'\, \hat{\phi}'}{a}$$

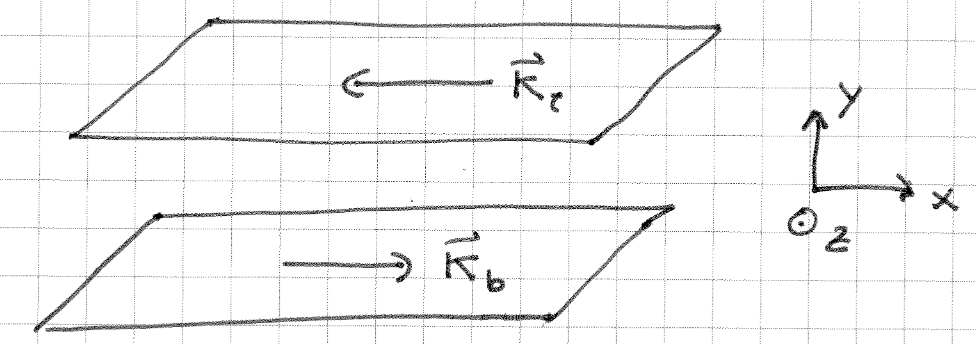
$$= \frac{\mu_0 I}{4\pi}\int_{\pi}^{2\pi} d\phi'\, \hat{\phi}'$$

$$= \frac{\mu_0 I}{4\pi}\int_{\pi}^{2\pi} d\phi' \left(-\sin\phi'\,\hat{x} + \cos\phi'\,\hat{y}\right)$$

$$= \frac{\mu_0 I}{4\pi} \cos\phi'\,\hat{x}\,\Big|_{\pi}^{2\pi}$$

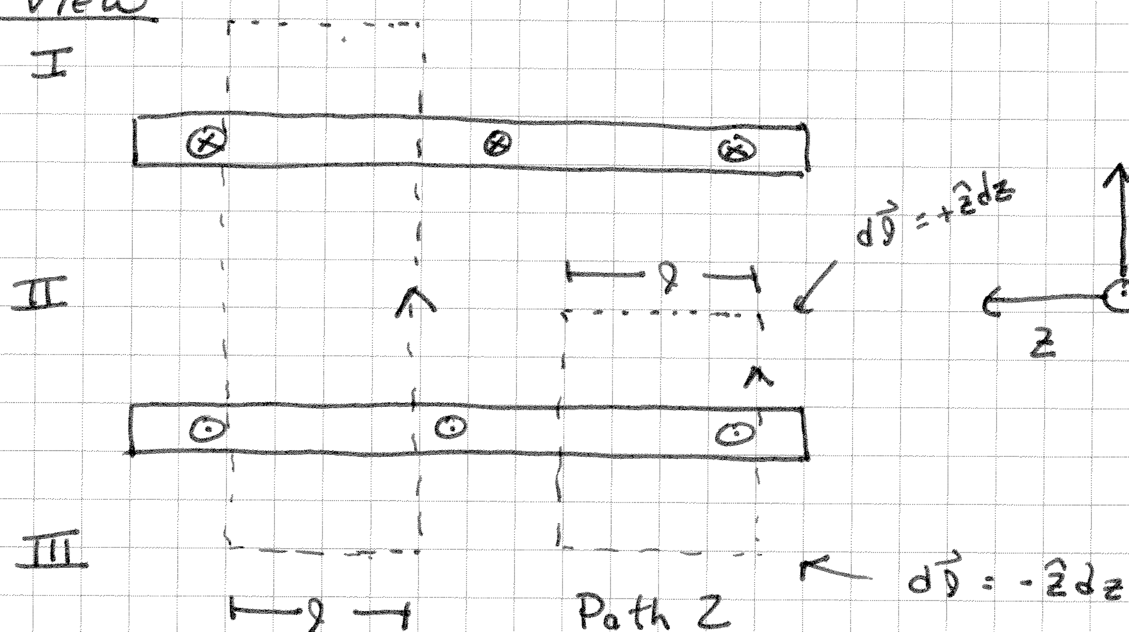
$$= \frac{\mu_0 I}{2\pi}\,\hat{x}$$

Ex Consider a system with two current sheets



$\vec{K}_t = -K_0 \hat{x}$ $\qquad$ $\vec{K}_b = K_0 \hat{x}$

End View



By RHR, normal $\hat{n} = +\hat{x}$ out of page for both paths.

<u>Path 1</u> $I_{enc} = -K_0 l + K_0 l = 0$

$$= \vec{K}_t \cdot \hat{n}\, l + \vec{K}_b \cdot \hat{n}\, l$$

By the fact that the fields do not decay with distance $\vec{B}_{III} = -\vec{B}_I$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = 0 = +B_I l - B_{III} l$$

$$\Rightarrow \vec{B}_I = \vec{B}_{III} = 0$$

note $+z$ to left

<u>Path 2</u> $I_{enc} = \vec{K}_b \cdot \hat{n}\, l = K_0 l$

Let $\vec{B}_I = B_I \hat{z}$ $\quad \vec{B}_{II} = B_{II} \hat{z}$ $\quad \vec{B}_{III} = B_{III} \hat{z}$

$$\oint \vec{B} \cdot d\vec{l} = \vec{B}_{III} \cdot (-\hat{z})\, l + \vec{B}_{II} \cdot (\hat{z})\, l$$

$$= \underbrace{-B_{III} l}_{=0} + B_{II} l = B_{II} l$$

$$B_{II} l = \mu_0 I_{enc} = \mu_0 K_0 l$$

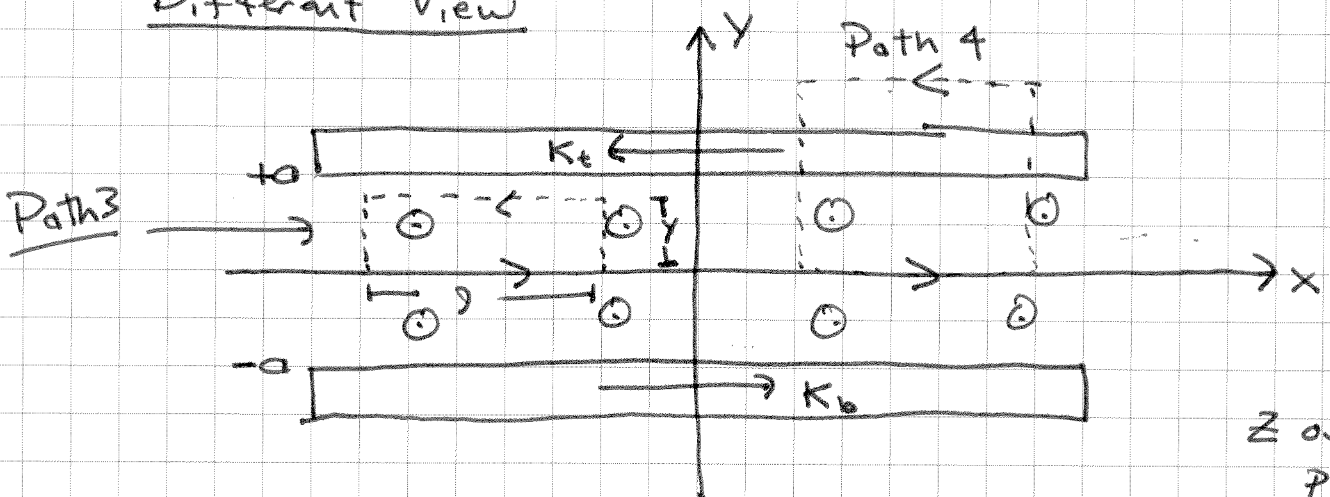
$$\vec{B}_{II} = \mu_0 K_0 \hat{z}$$

(Direction consistent with right hand rule)

Compute Vecter Potential

Let the origin be a point halfway between planes

Different View



z out of page

Choose $\vec{A}(0,0,z) = 0$ which you can because it is a potential.

Outward normal for both paths $\hat{n} = \hat{z}$ is out of the page.

Path 3 $\quad \Phi_m = \int_S \vec{B} \cdot d\vec{a} = B_{II}\, y\, \ell$

$$= \oint_C \vec{A} \cdot d\vec{\ell} = \vec{A}(x,0,z) \cdot \hat{x}\, \ell - \vec{A}(x,y,z) \cdot \hat{x}\, \ell$$

$$\vec{A}(x,y,z) \equiv A_{II}\, \hat{x}$$

$$B_{II}\, y\, \ell = -A_{II}\, \ell$$

$$\vec{A}_{II} = -B_{II}\, y\, \hat{x} = -\mu_0 K_0 y\, \hat{x}$$

$\vec{A}$ is continuous so

$$\vec{A}_{III} = \vec{A}_{II}(-a) = \mu_0 K_0 a\, \hat{x}$$

$$\vec{A}_{I} = \vec{A}_{II}(a) = -\mu_0 K_0 a\, \hat{x}$$

<u>Check the field is recovered</u>.

$$\nabla \times \vec{A} = \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -B_{II}\, y & 0 & 0 \end{vmatrix}$$

$$= B_{II}\, \hat{z}$$