Vectors

Scalar - An object with magnitude but not direction. Example, temperature.

Vector - An object with magnitude and direction. Example - mass flow.

- Transforms the same way as the position vector $\vec{r} = (x, y, z)$ under 3D rotations.
- **Representing Vectors** $\vec{A}$

$$\vec{A} = (x, y, z) \quad \text{(preferred)}$$
$$= \langle x, y, z \rangle$$

$$\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \quad \text{(preferred)}$$
$$= A_x \hat{\imath} + A_y \hat{\jmath} + A_z \hat{k}$$

- **Unit Vectors** - $\hat{A}$ is a vector of unit length (no units) in the direction of $\vec{A}$

Example $\hat{x} = (1, 0, 0)$

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<u>Matrices</u> ($\overleftrightarrow{T}$) - Vectors are transformed using matrices.

$$\overleftrightarrow{T} = \begin{pmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

<u>Transformed Vector</u>

$$\vec{A}' = \overleftrightarrow{T}\,\vec{A}$$

<u>Rotation Matrix</u> To rotate a vector by an angle θ about the z-axis.

$$\overleftrightarrow{T}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If an object is a vector, then rotating it with $\overleftrightarrow{T}_\theta$ has the same effect as rotating the object in the real world.

Consider the vector $\vec{A} = (1m, 1m, 0)$

If we rotate such a vector by 45° in the real world, we would get $(0, \sqrt{2}m, 0) = \vec{A}'$

$$\overleftrightarrow{T}_{45^\circ} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{A}' = \overleftrightarrow{T}_{45} \vec{A} = (0, \sqrt{2}m, 0) \checkmark$$

$$\begin{pmatrix} A_x' \\ A_y' \\ A_z' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1m \\ 1m \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ \sqrt{2}\,m \\ 0 \end{pmatrix}$$

Not all 3-tuples are vectors. If we did the same thing with an inventory from a zoo,

$$\vec{A} = (2 \text{ lions}, 3 \text{ tigers}, 2 \text{ bears})$$

we would get

$$\vec{A} = \left(\frac{2 \text{ lions} - 3 \text{ tigers}}{\sqrt{2}}, \frac{2 \text{ lions} + 3 \text{ tigers}}{\sqrt{2}}, 2 \text{ bears} \right)$$

<u>Symmetry</u> - A symmetry of a system is a transformation that leaves the system unchanged.

$\Rightarrow$ Symmetry is often loosely used. To correctly invoke a symmetry, one should give the transformation that leaves the system unchanged.

<u>Example</u> - A linear object lying along the z-axis is unchanged by rotations $\overleftrightarrow{T}_\theta$ about the z-axis.

<u>Example</u> - A point object is invarient under any rotation about its center.

Other Symmetries

Reflection (through x-y plane)

$$\overleftrightarrow{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Inversion (Reflection through origin)

$$\overleftrightarrow{T} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Psuedo Vectors All vectors must behave well under rotation or they aren't vectors. Vectors that don't change sign under inversion are called psuedovectors.

Example $\vec{C} = \vec{A} \times \vec{B}$. Under inversion, $\vec{A} \to -\vec{A}$, $\vec{B} \to -\vec{B}$, so $\vec{C} \to \vec{C}$.

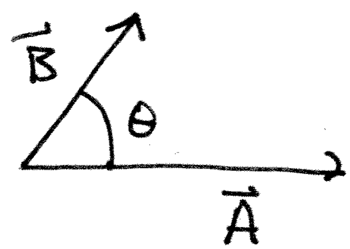
Vector Products

Dot Product

$$\vec{A} \cdot \vec{B} = \text{scalar}$$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

$$= |\vec{A}||\vec{B}| \cos\theta$$



Vector Modulus (Length)

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2} \equiv A$$

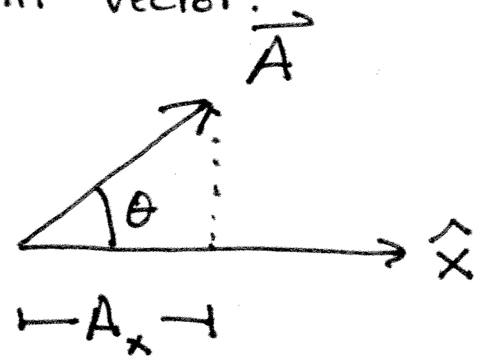
$$= \sqrt{\vec{A} \cdot \vec{A}}$$

Unit Vector

Vector of length 1 in the direction of $\vec{A}$.

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$$

Taking a dot product with a unit vector finds the projection of a vector in the direction of the unit vector.



$$A_x = \hat{x} \cdot \vec{A} = \underbrace{|\hat{x}|}_{=1}|\vec{A}|\cos\theta = A\cos\theta \ \checkmark$$

<u>Cross-Product</u> - The cross-product of $\vec{A}$ and $\vec{B}$ returns a vector $\perp$ to both $\vec{A}$ and $\vec{B}$.

$$\vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

determinant

$$|\vec{C}| = |\vec{A} \times \vec{B}| = AB\sin\theta$$

Properties of Cross-Product

- Does not commute

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

- $\vec{A} \times \vec{A} = 0$

- Direction - Right-hand Rule

  Point fingers of the right hand in the direction of $\vec{A}$, turn hand so fingers curl ($< 180°$) in the direction of $\vec{B}$, thumb points in the direction of $\vec{C}$

$$\underset{\text{thumb}}{\vec{C}} = \underset{\text{fingers}}{\vec{A}} \times \underset{\text{curl}}{\vec{B}}$$

- Cross-product does not associate.

$$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = 0$$

so

$$\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) = (\vec{A} \times \vec{B}) \times \vec{C}$$

$\Longrightarrow$ $\vec{A} \times \vec{B} \times \vec{C}$ has no meaning, need ( ).

Triple Products

$$\text{scalor} = \vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$$

$$= \det\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$\Rightarrow$ Volume of parallel-piped formed by $\vec{A}$, $\vec{B}$, $\vec{C}$.

BAC - CAB Rule

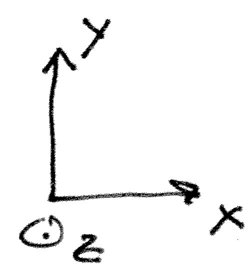
$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

vector multiplied by scalor

Proof Cross Product Does not Associate

$$(\hat{y}\times\hat{x})\times\hat{x} = (-\hat{z})\times\hat{x}$$

$$= -\hat{y}$$



$$\hat{y}\times(\hat{x}\times\hat{x}) = 0$$