EM Waves + Conductors

For Ohmic conductors (most)

$$\vec{J}_f = \sigma \vec{E}$$

and Maxwell's equations can be written

$$\nabla \cdot \vec{E} = \frac{\rho_f}{\varepsilon_0} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu_0 \sigma \vec{E} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Net free charge placed ~~on a~~ in a conductor tends to flow to the surface. The characteristic relaxation time is give by the continuity eqn

$$\nabla \cdot J_f = -\frac{\partial \rho_f}{\partial t} = \sigma \nabla \cdot \vec{E}$$

$$= -\frac{\sigma \rho_f}{\varepsilon}$$

$$\frac{\partial \rho_f}{\partial t} = -\frac{\sigma}{\varepsilon_0} \rho_f$$

$$\rho_f(t) = \rho_f(0)\, e^{-t/\tau}$$

$\tau = \varepsilon / \sigma$ relaxation time constant for free charge.

As an estimate assume $\varepsilon \sim \varepsilon_0$ and

$\sigma \sim 1 \times 10^{+8} \frac{1}{\Omega m}$ for a good conductor

$$\tau \sim \frac{8.85 \times 10^{-12}}{1 \times 10^{+8}} \sim 10^{-20}\,s$$

After a few τ, all the charge resides at the surface and we can set $\rho_f = 0$

Using the same procedure by which the wave equations were originally derived (a good test question) we find

$$\nabla^2 \vec{E} = \mu \varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu \sigma \frac{\partial \vec{E}}{\partial t}$$

$$\nabla^2 \vec{B} = \mu \varepsilon \frac{\partial^2 \vec{B}}{\partial t^2} + \mu \sigma \frac{\partial \vec{B}}{\partial t}$$

Again try a wave solution

$$\vec{E} = \vec{E}_0 e^{i(kz - \omega t)}$$

Look at the x-component

$$\nabla^2 E_{0x} e^{i(kz-\omega t)} = \mu \varepsilon \frac{\partial^2}{\partial t^2} E_{0x} e^{i(kz-\omega t)} + \mu \sigma E_{0x} e^{i(kz-\omega t)}$$

This equation is satisfied if

$$(ik)^2 = \mu\varepsilon(-i\omega)^2 + \mu\sigma(-i\omega)$$

$$-k^2 = -\mu\varepsilon\omega^2 - i\omega\mu\sigma$$

$$k^2 = \mu\varepsilon\omega^2 + i\omega\mu\sigma$$

Evidently k must now be complex.

To take the square-root, we have to write k^2 as

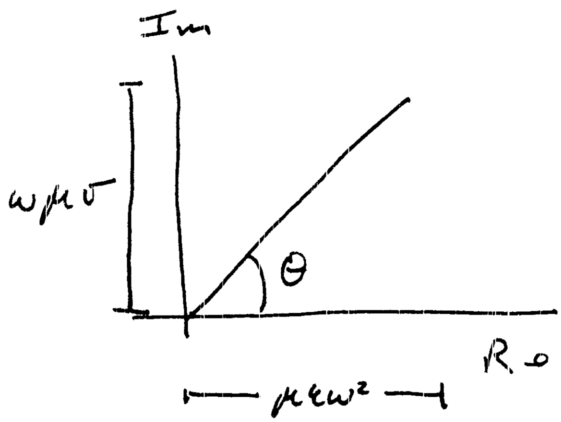
$$k^2 = |k^2|\, e^{i\theta}$$

$$|k^2|^2 = (k^2)(k^{2*}) = (\mu\varepsilon\omega^2 + i\omega\mu\sigma)(\mu\varepsilon\omega^2 - i\omega\mu\sigma)$$

$$= (\mu\varepsilon\omega^2)^2 + (\omega\mu\sigma)^2$$

$$= (\mu\omega)^2(\varepsilon^2\omega^2 + \sigma^2)$$

Find θ



$$\tan\theta = \frac{\omega\mu\sigma}{\mu\varepsilon\omega^2}$$

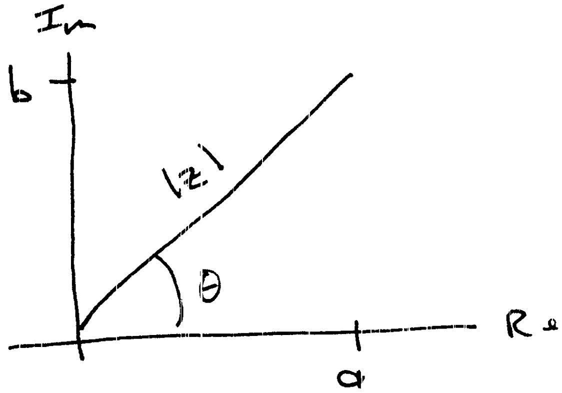
$$= \frac{\sigma}{\varepsilon\omega}$$

Square-root of a Complex Number

Consider an arbitrary complex number

$$z = |z| e^{i\theta} = a + ib \qquad (a, b \text{ real})$$

$$|z| = \sqrt{z^* z} = \sqrt{a^2 + b^2}$$



$$\tan\theta = b/a$$

The square root is $\sqrt{z} = \sqrt{|z|}\, e^{i\theta/2}$

$$= (a^2+b^2)^{1/4} e^{i\theta/2}$$

But what is that in terms of a and b?

$$e^{i\theta} = \frac{z}{|z|} = \underbrace{\frac{a}{\sqrt{a^2+b^2}}}_{\cos\theta} + i \underbrace{\frac{b}{\sqrt{a^2+b^2}}}_{\sin\theta}$$

$$e^{i\theta/2} = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}$$

Half-angle formulas

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}}$$

$$e^{i\theta/2} = \left(\frac{1+\frac{a}{|z|}}{2}\right)^{1/2} + i\left(\frac{1-\frac{a}{|z|}}{2}\right)^{1/2}$$

$$= \left(\frac{|z|+a}{2|z|}\right)^{1/2} + i\left(\frac{|z|-a}{2|z|}\right)^{1/2}$$

$$= \frac{1}{\sqrt{|z|}}\left(\frac{|z|+a}{2}\right)^{1/2} + i\left(\frac{|z|-a}{2}\right)^{1/2}$$

$$\sqrt{z} = \sqrt{|z|}\cdot e^{i\theta/2} = \left(\frac{|z|+a}{2}\right)^{1/2} + i\left(\frac{|z|-a}{2}\right)^{1/2}$$

That was interesting, now apply it.

$$K^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$$

$$= \mu\varepsilon\omega^2\left(1 + i\Gamma\right) \qquad \Gamma \equiv \frac{\sigma}{\varepsilon\omega}$$

$$\sqrt{K^2} = \sqrt{\mu\varepsilon\omega^2}\,\sqrt{1+i\Gamma}$$

$$= \omega\sqrt{\mu\varepsilon}\,\sqrt{1+i\Gamma}$$

$$\sqrt{1+i\Gamma} = \left(\frac{r+1}{2}\right)^{1/2} + i\left(\frac{r-1}{2}\right)$$

$$r = \left(\left|\sqrt{1+i\Gamma}\right|\right) = \sqrt{1+\Gamma^2}$$

$$\sqrt{1+i\Gamma} = \left(\frac{\sqrt{1+\Gamma^2}+1}{2}\right)^{1/2} + i\left(\frac{\sqrt{1+\Gamma^2}-1}{2}\right)^{1/2}$$

$$\sqrt{K^2} = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right) + i\left(\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right)\right.$$

$= K$ (which is obviously complex)

The wave number k can be split into real and imaginary pieces

$$k = k_r + k_i i$$

<u>Real Part</u>

$$k_r = \omega \sqrt{\frac{\varepsilon\mu}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} + 1 \right)$$

<u>Imaginary Part</u>

$$k_i = \omega \sqrt{\frac{\varepsilon\mu}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2} - 1 \right) > 1$$

Substituting this decomposition into $\vec{E}$

$$\vec{E} = \vec{E}_0 \, e^{i(kz - \omega t)} = \vec{E}_0 \, e^{i[(k_r + ik_i)z - \omega t]}$$

$$= \vec{E}_0 \underbrace{e^{-k_i z}}_{\text{Decaying part}} \underbrace{e^{i(k_r z - \omega t)}}_{\text{oscillating part}}$$

$\Rightarrow$ The wave decays (attenuates) as it travels through the material.

A wave striking a conducting surface therefore penetrates some distance into the surface.

The characteristic depth of penetration is called the <u>skin depth</u>, d.

$$d = \frac{1}{k_i}$$

<u>Ex</u> Let's work on popping popcorn. I would guess the Tesla coil runs at a frequency about $f = 1000$ Hz. Popcorn should have $\varepsilon \sim \varepsilon_0$ and $\mu \sim \mu_0$. Guess the conductivity of popcorn $\sim 10^5 \frac{1}{\Omega m}$ about that of carbon.

$$k_i = 2\pi f \left(\frac{\varepsilon_0 \mu_0}{2}\right)^{1/2} \left[\sqrt{1 + \left(\frac{\sigma}{2\pi f \varepsilon_0}\right)^2} - 1\right]$$

$$= \frac{\sqrt{2}\pi f}{c} \left[\sqrt{1 + \underbrace{\left(\frac{10^5}{2\pi 10^3 \cdot 8.85\times 10^{-12}}\right)^2}_{10^{24}}} - 1\right]$$

$$\simeq \frac{\sqrt{2}\pi f}{c} \cdot \frac{\sigma}{2\pi f \varepsilon_0} \equiv \frac{\sigma}{\varepsilon_0 c} = \frac{10^5}{(8.85\times 10^{-12})(3\times 10^8)}$$

$$\approx 10^9 \text{ m}^{-1}$$

Skin depth $\quad d = \frac{1}{k_i} \sim 1\,\text{nm}$

$\Rightarrow$ No penetration

$$\vec{E} = E_0 e^{-k_i z} e^{i(kz - \omega t)} \hat{x} = E_0 e^{i(kz-\omega t)} \hat{x}$$

complex

$\Rightarrow$ Assume $\hat{n} = \hat{x}$

Apply Faradays Law as before

$$\vec{B} = \frac{k}{\omega} E_0 e^{i(kz-\omega t)} \hat{y}$$

the same as before but this time k is complex,

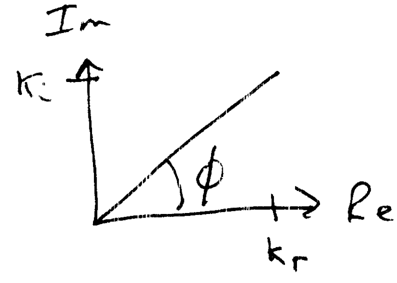
so $\quad \frac{k}{\omega} \neq c$

Since k is complex, there is a phase difference between the electric and magnetic components.

As with any complex number,

$$k = |k| e^{i\phi}$$

$$|k| = \sqrt{k_r^2 + k_i^2}$$



$$\tan\phi = \frac{k_i}{k_r}$$

If we extract the phases from E_0:

$$E_0 = |E_0| e^{i\delta_E}$$

then
$$\vec{E} = |E_0| e^{i(kz - \omega t + \delta_E)} \hat{x}$$

$$\vec{B} = \frac{|E_0||k|e^{i\phi}}{\omega} e^{i(kz-\omega t+\delta_E)} \hat{y}$$

$$= |B_0| e^{i(kz-\omega t+\delta_E+\phi)} \hat{y}$$

$\Rightarrow$ The magnetic component now lags the electric component (gets to the origin later) by ϕ.

$\Rightarrow$ The ratio of the electric to magnetic component is changed

$$\frac{|B_0|}{|E_0|} = \frac{|k|}{\omega} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}$$