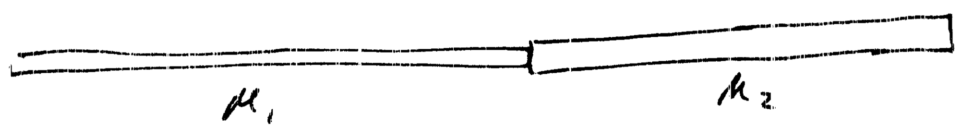
General Wave Equation

Consider a system that satisfies the wave equation with v_1 for $x<0$ and v_2 for $x>0$. For example, two strings with mass density μ_1 and μ_2 connected at the origin.



$$v_1 = \sqrt{\frac{T}{\mu_1}} \qquad v_2 = \sqrt{\frac{T}{\mu_2}}$$

T = tension

A wave is setup ~~for~~ at $-\infty$ travelling to the right, the incidant wave

$$f_i(x,t) = A_i \cos(k_1 x - \omega t)$$

$$f_i(x,t) = \mathrm{Re}\left(\tilde{f}_i(x,t)\right) = \mathrm{Re}\left(A_i e^{i(k_1 x - \omega t)}\right)$$

At the point the two strings join, some of the wave is reflected as

$$\tilde{f}_r(x,t) = \tilde{A}_r e^{i(-k_1 x - \omega t)}$$

and some of the wave is transmitted

$$\tilde{f}_t(x,t) = \tilde{A}_t e^{i(k_2 x - \omega t)}$$

Note, we chose the frequency of the incoming wave and for the point $x = 0$ to stay connected both the reflected and transmitted wave must share that frequency.

Boundary Conditions

The boundary conditions must be determined from the physics of the system from which the wave equation was derived. For the string, f must be continuous so the string does not come apart and $T\frac{\partial f}{\partial x}$, the force on the string must be continuous so no point has infinite acceleration.

f continuous at $x=0$

$$f_i(0,t) + f_r(0,t) = f_t(0,t)$$

$$A_i e^{-i\omega t} + \tilde{A}_r e^{-i\omega t} = \tilde{A}_t e^{-i\omega t}$$

$$A_i + \tilde{A}_r = \tilde{A}_t \qquad (1)$$

$\frac{\partial f}{\partial x}$ continuous at $x = 0$

$$i k_1 A_i - i k_1 \tilde{A}_r = i k_2 \tilde{A}_t$$

$$A_i - \tilde{A}_r = \frac{k_2}{k_1} \tilde{A}_t \qquad (2)$$

$$(1) + (2) \Rightarrow \quad 2 A_i = \left(1 + \frac{k_2}{k_1} \right) \tilde{A}_t$$

$$\frac{\tilde{A}_t}{A_i} = \frac{2 k_1}{k_1 + k_2}$$

$$\cancel{(2) - \frac{k_2}{k_1}(2) \Rightarrow}$$

$$A_i + \tilde{A}_r = \tilde{A}_t = \frac{2 k_1}{k_1 + k_2} A_i$$

$$\frac{\tilde{A}_r}{A_i} = \frac{2 k_1}{k_1 + k_2} - 1 = \frac{k_1 - k_2}{k_1 + k_2}$$

Note, we only solve for the ratio of the amplitude. We can write these in terms of the wave velocities -

$$v_1 = \frac{\omega}{k_1} \qquad v_2 = \frac{\omega}{k_2}$$

$$\frac{\tilde{A}_r}{A_i} = \frac{\cancel{2} v_2 - v_1}{v_2 + v_1} \qquad \frac{\hat{A}_t}{A_i} = \frac{2v_2}{v_1 + v_2}$$

The complex amplitudes contain the phase shifts

$$\tilde{A}_r = |\tilde{A}_r| e^{i\delta_r} \qquad \hat{A}_t = |\hat{A}_t| e^{i\delta_t}$$

The transmitted wave always has $\delta_t = 0$, so the transmitted wave is in phase with the incident wave.

If $v_2 > v_1$, the reflected wave is also in phase with $\delta_r = 0$.

If $v_2 < v_1$ then

$$\tilde{A}_r = (-1)\left(\frac{v_1 - v_2}{v_1 + v_2}\right) = \left|\frac{v_1 - v_2}{v_1 + v_2}\right| e^{i\pi}$$

$$\delta_r = \pi$$

The reflected wave is ~~top~~ flipped over with a phase shift of π.

For a string $v = \sqrt{\frac{T}{\mu}}$ and $v_2 < v_1 \Rightarrow \mu_2 > \mu_1$ so the phase shift occurs when string 2 is heavier.