# Lecture 10 - 2/10/2003

# Section 1 - Generalize To 3D

## Newton's Second Law

$$\vec{F} = \frac{d\vec{p}}{dt}$$

## Power (Time Rate of Change of T)

$$\vec{F} \cdot \vec{v} = \frac{d\vec{p}}{dt} \cdot \vec{v} = \frac{d}{dt}\left(\frac{1}{2} m \vec{v} \cdot \vec{v}\right) = \frac{dT}{dt}$$

Kinetic Energy

$$T = \frac{1}{2} m \vec{v} \cdot \vec{v}$$

## Work

$$\int_{A \to B} \vec{F} \cdot d\vec{r} = \int_{A \to B} dT = T_B - T_A$$

since $\vec{F} \cdot \vec{v} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \frac{dT}{dt}$

Potential - If F conservative, a potential function exists

$$\vec{F} = -\nabla V$$

$$\int_{A \to B} \vec{F} \cdot d\vec{r} = -\int_{A \to B} \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$= -V(B) + V(A) = T(B) - T(A)$$

$$T(A) + V(A) = T(B) + V(B) = \underline{\text{Mechanical Energy}}$$

# Section 2 - Potential Functions

Grad $\nabla$ - $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

curl $f$ = $\nabla \times \vec{F}$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

For a force to be conservation, the potential energy must return to ~~its~~ the same value each time a particle is returned to point A.

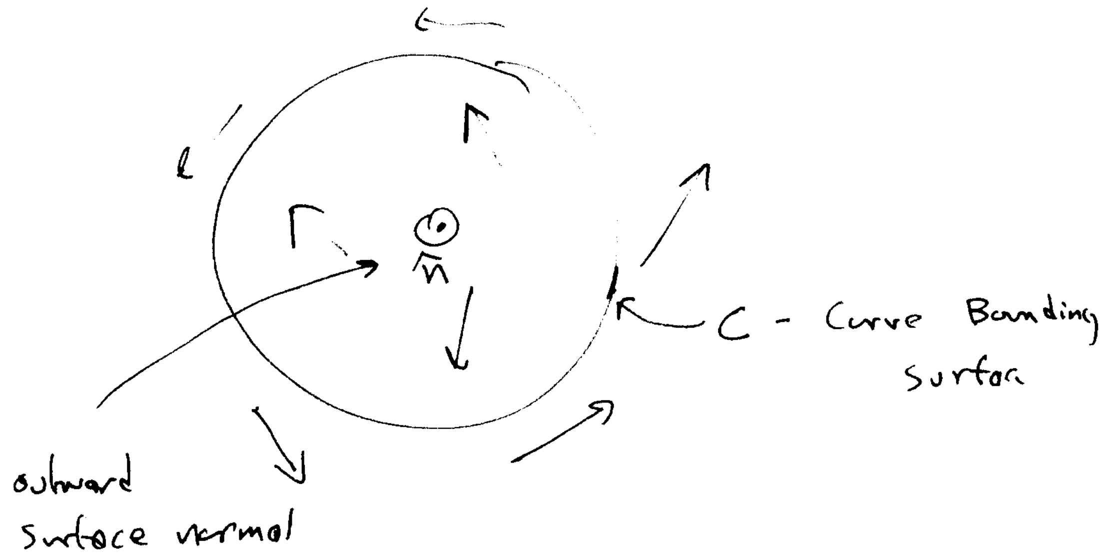
$\Rightarrow$ Independance of path - The potential difference $\Delta V_{AB} = V(B) - V_A$ must be the same along any path

$\Rightarrow$ The potential difference around any closed loop is zero.

## Stoke's Thm

$$\oint_C \vec{F} \cdot d\vec{r} = \int_S (\nabla \times \vec{F}) \cdot \hat{n} \, dA$$

closed path




## UP II Students

**Ampere's Law**

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I = \oint_S (\nabla \times \vec{B}) \cdot \hat{n} \, dA$$

$$\nabla \times \vec{B} = \mu_0 \frac{I}{A} + \text{Displacement}$$

surface area becomes small.

So for the work done to be independant of path $\boxed{\nabla \times \vec{F} = 0}$

<u>Condition for Conservative Force</u>

$$\nabla \times \vec{F} = 0$$

$\Rightarrow$ The potential is additive, since

$$\vec{F}_{net} = \sum F = -\frac{d}{dx}\sum U(x)$$

you can add potential functions.

(But only choose ~~one~~ constant once)

# Section 3 - Examples

## Conservative Force Field

$$\vec{F} = -\frac{GmM}{r^2}\,\hat{e}_r$$

$$\vec{F} = \frac{kq_1q_2}{r^2}\,\hat{e}_r$$

spherical

## Non-Conservative

$$\vec{F} = -\mu^k F^n = -\mu_k mg\,O \quad \text{Friction}$$

$$\vec{F} = \cancel{\frac{\mu_0 I}{2\pi r}\,\hat{e}_\theta}$$

$$= \frac{BAq}{2\pi r}\,\hat{e}_\phi$$

Polar

Charge in ~~uniformly~~ linear ~~in~~ increasing electric field, cylindrical region.

$$= \frac{B\pi r^2 q}{2\pi r}\,\hat{e}_\phi = \frac{Bqr}{2}\,\hat{e}_\phi$$

Hold the phone - Frictional Force is constant, why isn't that conservative. $\Rightarrow$ Because the direction depends on velocity

$$\vec{F} = -\mu_k mg\,\hat{v} = -\mu_k mg\,\frac{\vec{v}}{v}$$

How do we take curl in spherical and polar coordinates to prove the force is conservation or non-conservative.

Appendix F

$$\nabla \times \vec{F} = \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

Cylindrical $\quad r, \phi, z \qquad h_r = 1, \; h_\phi = r, \; h_z = 1$

$$\nabla \times \vec{F} = \frac{1}{r} \begin{vmatrix} \hat{e}_r & r\hat{e}_\phi & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_r & rF_\phi & F_z \end{vmatrix}$$

What is curl $\quad \vec{F} = \frac{B\cancel{A}qr}{2} \hat{e}_\phi$

$$\nabla \times \vec{F} = \frac{1}{r} \begin{vmatrix} \hat{e}_r & r\hat{e}_\phi & e_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & r\left(\frac{Bqr}{2}\right) & 0 \end{vmatrix} = Bqr\,\hat{e}_r \neq 0$$

from pole

$r, \theta, \phi$

Spherical — $h_r = 1$, $h_\theta = r$, $h_\phi = r \sin\theta$

$$\vec{F} = \alpha \frac{\hat{e}_r}{r^2} \qquad ( \alpha = k q_1 q_2 \ \text{or} \ \alpha = GmM$$

$$\nabla \times \vec{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\, \hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r F_\theta & r\sin\phi\, F_\phi \end{vmatrix}$$

if $\vec{F} = \dfrac{\alpha \hat{e}_r}{r^2}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\, \hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{\alpha}{r^2} & 0 & 0 \end{vmatrix}$$

$$= 0$$

$\Rightarrow$ Gravitational + Electric Forces are conservative, we live.

# Lecture - 2/12/2003

- ~~Mary's Test Score~~ –

# Homework Notes

(4.9) Find trigonometric condition for maximum height and range. Do ~~not do~~ numeric calculation, with

$$\theta_{max,\,height} = 70.5^\circ$$

$$\theta_{max,\,range} = 39^\circ$$

(4.14) By numerically, they mean mathematically.

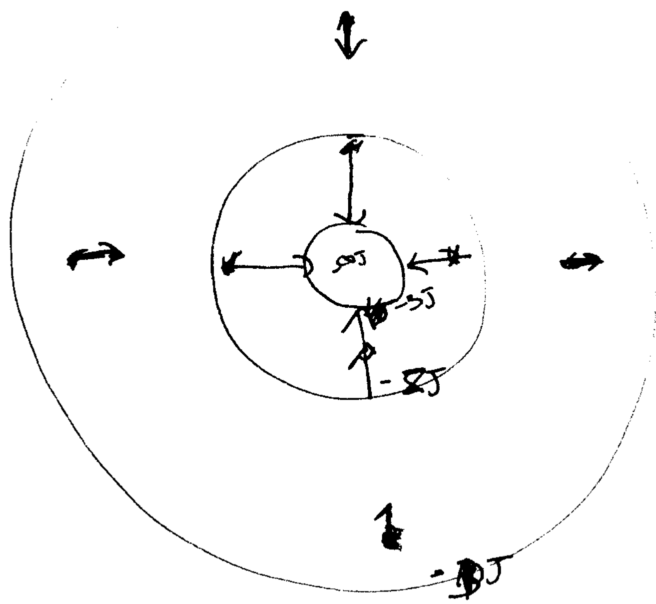
(4.17) Long but worth it.

# Section 4 - Representing Potential

## Contour Maps

$$V = \frac{-k q_1 q_2}{r}$$

- Draw with lines of constant potential energy - equipotentials




~~If $q_1 > 0$~~

Where does a mass go if released $\Rightarrow$

Lowest Potential Energy

$\Rightarrow$ Force points in direction of ~~increasing~~ decreasing potential.

> A well drawn contour map will have lines with equal energy spacing.

$\Rightarrow$ Force stronger where equipotentials closer together

# Section 2 - Work

**Problem** A $+q$ charge is moved from ~~(0, a, 0)~~

$\vec{A} = (0, a, 0)$ to $\vec{B}(a, a, 0)$ in the field of a $-q$ charge.

$F = -\frac{kq^2}{r^2}\hat{r}$ on a surface with coefficient of

kinetic friction $\mu^k$. Calculate total work done by

external force. The charge begins and ends at rest.

$$W = \Delta T + \Delta U + E_{diss}$$

$$\Delta T = 0$$

friction force always in same direction as motion

$$E_{diss} = \int \vec{F}_{diss} \cdot d\vec{r} = \int_{\vec{A} \to \vec{B}} \mu_k mg \, ds = \mu_k mg a$$

$$\vec{F} = -\frac{kq^2}{r^2}\hat{r} = -\nabla U = -\left[\hat{e}_r \frac{\partial}{\partial r} + \underbrace{\frac{\hat{e}_\theta}{r}\frac{\partial}{\partial \theta}}_{0} + \underbrace{\frac{\hat{e}_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}}_{0}\right] U$$

$$\frac{kq^2}{r^2} = \frac{\partial U}{\partial r}$$

$$U(r) = \int \frac{kq^2}{r^2} dr = -\frac{kq^2}{r} + C$$

$$U(\infty) = 0 \Rightarrow C = 0$$

$$\Delta U = U(\vec{B}) - U(\vec{A}) = U(\cancel{2a}\,\sqrt{2}a) - U(a)$$

$$= \frac{-kq^2}{\sqrt{2}\,a} + \frac{kq^2}{a} > 0$$

$$W = 0 - \frac{kq^2}{\sqrt{2}\,a} + \frac{kq^2}{d} + \mu_k m g a$$

# Section 3 - Let things Move

$$\vec{F} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2}$$

$$m\ddot{x} = f_x(\vec{r}, \vec{v})$$

$$m\ddot{y} = f_y(\vec{r}, \vec{v})$$

$$m\ddot{z} = f_y(\vec{r}, \vec{v})$$

3 differential equations

$\Rightarrow$ 3 sets of initial condition,

$x(0)$ $\dot{x}(0)$

$y(0)$ $\dot{y}(0)$

$z(0)$ $\dot{z}(0)$

I. Solve each eqn $\Rightarrow$ 2 constants / eqn

$\Rightarrow$ 6 constants total

II. Use initial conditions to fix constants

$\boxed{\text{This could get bad}}$

Lecture 2/14/2003

- Make problem 4.5 bonus.
- Series Expansion - $\frac{1}{1+x} = 1 - x + x^2 + \cdots$
- 4.17 - $\boxed{\text{Diff E}}$

$$\ddot{x} + \omega_0^2 x = \alpha t$$

$$x(t) = x_h(t) + x_p(t)$$

$$x_h(t) = A_1 \cos\omega_0 t + A_2 \sin\omega_0 t$$

Guess $x_p(t) = C_1 t + C_2$

$$\omega_0^2 C_1 t + C_2 \omega_0^2 = \alpha t \quad \text{for all } t$$

$$\Rightarrow C_2 = 0$$

$$C_1 = \alpha / \omega_0^2$$

# Section 4 - Separable Forces

Things are must simplier if the force is such that

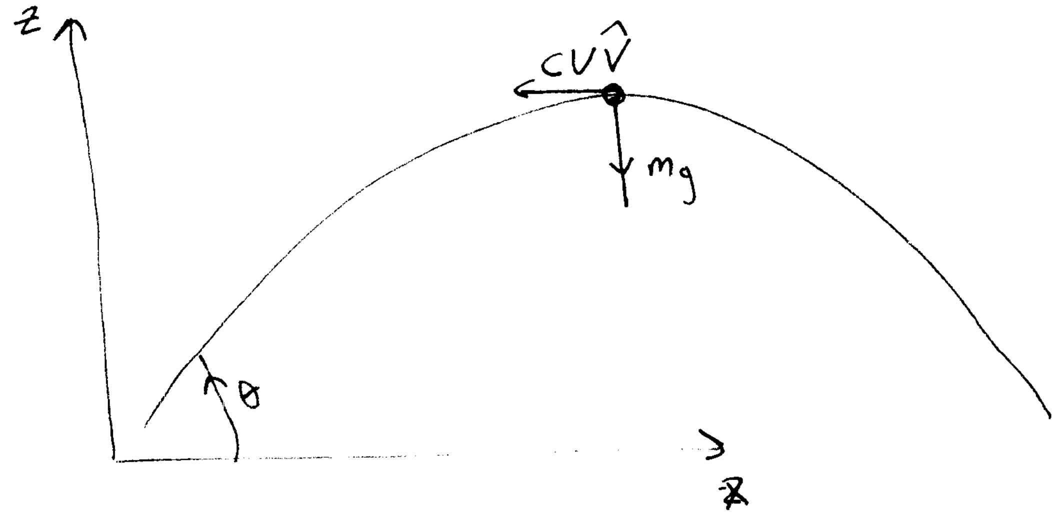
$$m\ddot{x} = f_x(x, \dot{x})$$

$$m\ddot{y} = f_y(y, \dot{y})$$

$$m\ddot{z} = f_z(z, \dot{z})$$

This gives three independant equations of motion that can be solved using techniques of chapter 2.

<u>Example</u> - Flight of baseball experiencing a linear drag force.



Luckily, $c v \hat{v} = c\vec{v} = c(v_x, v_y, v_z)$

$$\vec{F} = -mg\hat{k} - c\vec{v}$$

$$= (-mg - cv_z)\hat{k} - cv_x\hat{i} - cv_y\hat{j} = m\frac{d\vec{v}}{dt}$$

EOM

$$m\ddot{x} = -c\dot{x}$$

$$m\ddot{y} = -c\dot{y}$$

$$m\ddot{z} = -mg - c\dot{z}$$

Initial Conditions

$$x(0) = 0 = y(0) = z(0)$$

$$\dot{x}(0) = V_{ox} \qquad \dot{y}(0) = 0 \qquad \dot{z}(0) = V_{oz}$$

$$\vec{V}_o = (V_{ox}, 0, V_{oz}) = V_o(\cos\theta, 0, \sin\theta)$$

$$V_{ox} = V_o\cos\theta \qquad V_{oz} = V_o\sin\theta$$

Solve $\ddot{x} = -\frac{c}{m}\dot{x}$

$$\frac{dv}{dt} = -\frac{c}{m} v \qquad \text{exponential decay}$$

$$v(t) = v_0 e^{-c/m\, t}$$

$$x(t) = \int_0^t v(t)\,dt = -\frac{mv_0}{c} e^{-\frac{ct}{m}} \Big|_0^t$$

$$= \frac{mv_{0x}}{c}\left(1 - e^{-\frac{ct}{m}}\right)$$

~~Turning points? Terminal Velocity?~~

~~No Maximum Height, Range~~

~~Max. Height $v_z = 0$~~

~~Time to reach max is~~

$x(\infty)$ is the maximum possible range, but may not be the maximum range.

Why?.

Solve $\quad \ddot{y} = 0 \qquad y(0) = 0 \quad \dot{y}(0) = 0$

$$y(t) = 0$$

Solve $\quad m\ddot{z} = -mg - cv_z$

$$\ddot{z} = -g - \frac{c}{m}\dot{z} = \frac{d\dot{z}}{dt}$$

$$\int_0^t dt = -\int_{v_{0z}}^{v_z} \frac{dv_z}{g + \frac{c}{m}v_z} \qquad u = g + \frac{c}{m}v_z \quad du = \frac{c}{m}dv_z$$

$$t = -\frac{m}{c}\int_{u_0}^{u} \frac{du}{u}$$

$$-\frac{ct}{m} = \ln\left(u/u_0\right)$$

$$u = u_0 e^{-ct/m}$$

$$g + \frac{c}{m}\dot{z} = \left(g + \frac{c}{m}v_{z0}\right)e^{-ct/m}$$

$$\dot{z} = \left[\frac{gm}{c} + v_{z0}\right]e^{-ct/m} - \frac{mg}{c}$$

$$= v_{oz}e^{-ct/m} - \frac{mg}{c}\left[1 - e^{-ct/m}\right]$$

$$= \frac{dz}{dt}$$

$$\int_0^z dz = z = \int_0^t v_{oz}\, e^{-ct/m} dt - \frac{mg}{c}\int_0^t \left(1 - e^{-ct/m}\right) dt$$

$$= \frac{m v_{oz}}{c}\left(1 - e^{-ct/m}\right) - \frac{mg}{c} t$$

$$+ \frac{m^2 g}{c^2}\left(1 - e^{-ct/m}\right)$$

$\Rightarrow$ In this problem, we fixed the ~~constants~~ initial conditions by using definite limits of integration

$\Rightarrow$ This still works, $m\ddot{z} = m\frac{d\dot{z}}{dt} = m\dot{z}\frac{d\dot{z}}{dz}$

just like $m\frac{dv}{dt} = mv\frac{dv}{dx}$ ~~if~~ $v\cancel{(x,t)}$]

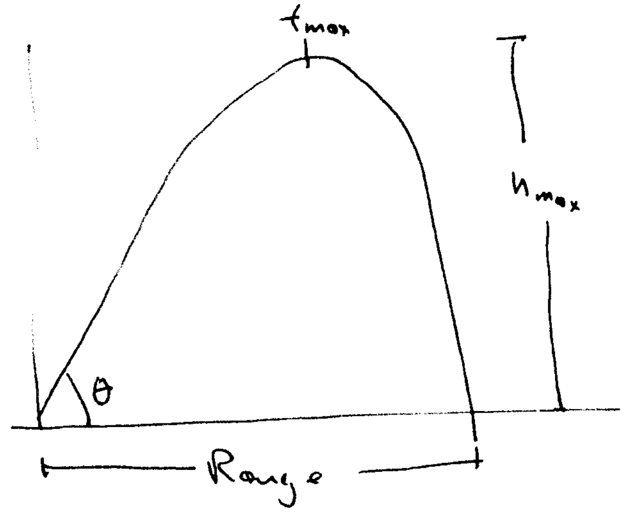
What can we find from trajectory?

Time to max hieght ($t_{max}$)

Max height ($h_{max}$)

Range ($R$)

Time of flight ($t_f$)




Note $t_f \neq 2t_r$

At $t_{max}$, $\quad V_z(t_{max}) = 0$

$$\dot{z}(t_{max}) = 0 = V_{0z}\, e^{-ct/m} - \frac{mg}{c}\left[1 - e^{-ct/m}\right]$$

$$\frac{mg}{c} = \left[V_{0z} + \frac{mg}{c}\right] e^{-ct/m}$$

$$\ln\left[\frac{\frac{mg}{c}}{V_{0z} + \frac{mg}{c}}\right] = -\frac{c}{m}\, t_{max}$$

$h_{max} = z(t_{max})$ Mess.

Time of Flight ($t_f$)

~~Range~~ - $z(t_f) = 0$

Range - $x(t_f) = R$

All of which we would be better served to find numerically.

Road Map $\vec{F} = m\vec{a} \Rightarrow$ 3 eqns / 6 constants

If separable, solve each independently.

# Section 5 - Non-separable Forces

**Friction** $$\vec{F} = -\mu^k m g \, \hat{v} = -\mu^k m g \left[ \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}, \frac{\dot{y}}{\quad}, \dot{z} \right]$$

**Quadratic Drag**

$$\vec{F} = -c_2 v^2 \hat{v} = -c_2 \left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)^{1/2} \vec{v}$$

$$= -c_2 v \vec{v}$$

Electricity + **Magnetism**

separable

non-separable.

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$$

**Example** $$\vec{F} = -\dot{y}\,\hat{\imath} + \dot{x}\,\hat{\jmath} +$$

**EOM**

$$m\ddot{x} = -\dot{y}$$

$$m\ddot{y} = \dot{x}$$

**Initial Conditions**

$$x(0) = y(0) = 0$$

$$\dot{x}(0) = 0$$

$$\dot{y}(0) = v_0$$

Integrate each eqn

$$m \int \frac{d\dot{x}}{dt}\, dt = -\int \frac{dy}{dt}\, dt$$

$$m\dot{x} = -y + C_1$$

Initial Condition

$$m\dot{x}(0) = 0 = -y(0) + C_1 = 0 + C_1$$

$$C_1 = 0$$

$$m\dot{x} = -y$$

Solve other eqn

$$m \int \frac{d\dot{y}}{dt}\, dt = \int \frac{dx}{dt}\, dt$$

$$m\dot{y} = x + C_2$$

Initial Condition

$$m\dot{y}(0) = mV_0 = x(0) + C_2 = C_2$$

$$C_2 = mV_0$$

$$m\dot{y} = x + mV_0$$

Separate the Equations

$$m \cancel{\ddot{y}} = \cancel{\dot{x}} \qquad \text{but} \qquad \dot{x} = -\frac{1}{m} y$$

$$m \ddot{y} = -\frac{1}{m} y$$

$$\ddot{y} + \frac{1}{m^2} y = 0 \qquad \omega_{oy}^2 = \frac{1}{m^2}$$

$$y(t) = A \cos(\omega_{oy} t + \phi)$$

$$= A_1 \cos \omega_{oy} t + A_2 \sin \omega_{oy} t$$

Likewise

$$m \ddot{x} = -\dot{y} = -\frac{1}{m}(x + \cancel{m}v_0) = -\frac{1}{m}(x + mv_0)$$

$$\ddot{x} + \frac{1}{m^2} x = \frac{-v_0}{m}$$

Move DifE

solves homogeneous eqn

$$\text{Solution} \qquad x(t) = x_h(t) + x_p(t)$$

solve non-homogeneous part

Guess $x_p(t) = C$

$$\ddot{x}_p + \frac{1}{m^2} x_p = \frac{C}{m^2} = -\frac{v_0}{m} \implies C = -mv_0$$

$$x(t) = A_x \cos(\omega_0 t + \phi_x) - mv_0$$

$$= A_{x1} \cos(\omega_0 t) + A_{x2} \sin \omega_0 t - mv_0$$

Fix $A_x, A_y, \phi_x, \phi_y$ using initial conditions.

May be more convienient to write -

$$x(t) = A_{x1} \cos \omega_0 t + A_{x2} \sin \omega_0 t - mv_0$$

$$x(0) = A_{x1} - mv_0 = 0 \Rightarrow A_{x1} = mv_0$$

$$\dot{x}(t) = -\omega_0 A_{x1} \sin \omega_0 t + A_{x2} \omega_0 \cos \omega_0 t$$

$$\dot{x}(0) = 0 = A_{x2} \omega_0 \Rightarrow A_{x2} = 0$$

$$\boxed{x(t) = mv_0 \cos \omega_0 t - mv_0}$$

$$y(t) = A_{y1} \cos \omega_0 t + A_{y2} \sin \omega_0 t$$

$$y(0) = 0 \Rightarrow A_{y1} = 0$$

$$\dot{y}(t) = +\omega_0 A_{y2} \cos \omega_0 t =$$

$$\dot{y}(0) = v_0 = \omega_0 A_{y2}$$

$$A_{y2} = \frac{v_0}{\omega_0} \qquad y(t) = \frac{v_0}{\omega_0} \sin \omega_0 t$$