Section 3.1 - Basics

Our game is to compute trajectories, the position of a particle as a function of time.

Position

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$$

$$\hat{\imath} \times \hat{\jmath} = \hat{k}$$

Velocity

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \dot{x}\,\hat{\imath} + \dot{y}\,\hat{\jmath} + \dot{z}\,\hat{k}$$

$$\dot{x} \equiv \frac{dx}{dt}$$

Acceleration

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \ddot{x}\,\hat{\imath} + \ddot{y}\,\hat{\jmath} + \ddot{z}\,\hat{k}$$

Section 3.2 - Vector Tricks

Dot Product — $\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$

$$= |A||B|\cos\theta$$

If $\vec{A}\perp\vec{B} \Rightarrow \vec{A}\cdot\vec{B} = 0$

Cross Product — $$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$|\vec{A}\times\vec{B}| = |\vec{A}||\vec{B}|\sin\theta$$

Scaler Triple Product $$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & \cdots & \\ B_z & \cdots & \end{vmatrix}$$

$$= (\vec{A}\times\vec{B})\cdot\vec{C}$$

Vector Triple Product (BAC - CAB)

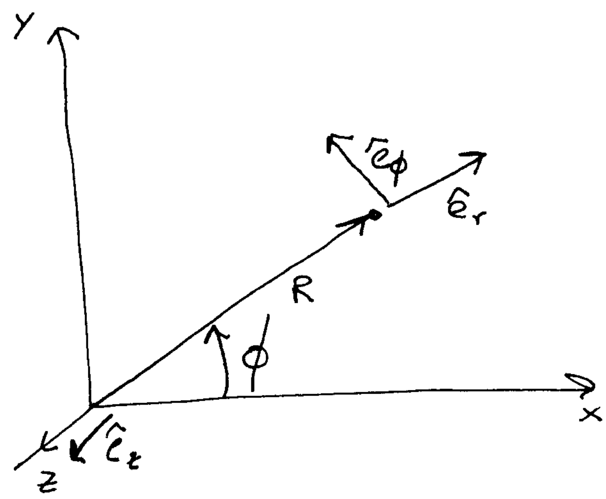
$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

$$\neq (\vec{A}\times\vec{B})\times\vec{C}$$

Non-associative

Section 3.3 – Polar Coordinates

Describe location of a particle using the variables, r, ϕ, z



$R \equiv$ Distance from z-axis
$\phi \equiv$ Angle about z-axis

Unit Vectors $\quad \hat{e}_r \times \hat{e}_\phi = \hat{e}_z$

Position Vector

$$\vec{r}(t) = R\hat{e}_r + z\,\hat{e}_z = R\cos\phi\,\hat{\imath} + R\sin\phi\,\hat{\jmath} + z\,\hat{k}$$

Construct Unit Vectors

$$\hat{e}_r = \cos\phi\,\hat{\imath} + \sin\phi\,\hat{\jmath}$$

$\hat{e}_\phi \perp \hat{e}_r$ – By observation $\quad \hat{e}_\phi \cdot \hat{e}_r = 0$

$$\hat{e}_\phi = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}$$

* sign chosen so sense of vector correct for diagram.

Velocity

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dR}{dt}\hat{e}_r + R\frac{d\hat{e}_r}{dt} + \dot{z}\,\hat{e}_z$$

$$\frac{d\hat{e}_r}{dt} = -\dot{\phi}\sin\phi\,\hat{\imath} + \dot{\phi}\cos\phi\,\hat{\jmath} = \dot{\phi}\,\hat{e}_\phi$$

$$\boxed{\vec{v}(t) = \dot{R}\,\hat{e}_r + R\dot{\phi}\,\hat{e}_\phi + \dot{z}\,\hat{e}_z}$$

Acceleration

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \ddot{R}\,\hat{e}_r + \dot{R}\frac{d\hat{e}_r}{dt} + \dot{R}\dot{\phi}\,\hat{e}_\phi + R\ddot{\phi}\,\hat{e}_\phi$$

$$+ R\dot{\phi}\frac{d\hat{e}_\phi}{dt} + \ddot{z}\,\hat{e}_z$$

$$\frac{d\hat{e}_\phi}{dt} = -\dot{\phi}\cos\phi\,\hat{\imath} - \dot{\phi}\sin\phi\,\hat{\jmath}$$

$$= -\dot{\phi}\,\hat{e}_r$$

$$\vec{a}(t) = \ddot{R}\,\hat{e}_r + \dot{\phi}\dot{R}\,\hat{e}_\phi + \dot{R}\dot{\phi}\,\hat{e}_\phi + R\ddot{\phi}\,\hat{e}_\phi$$

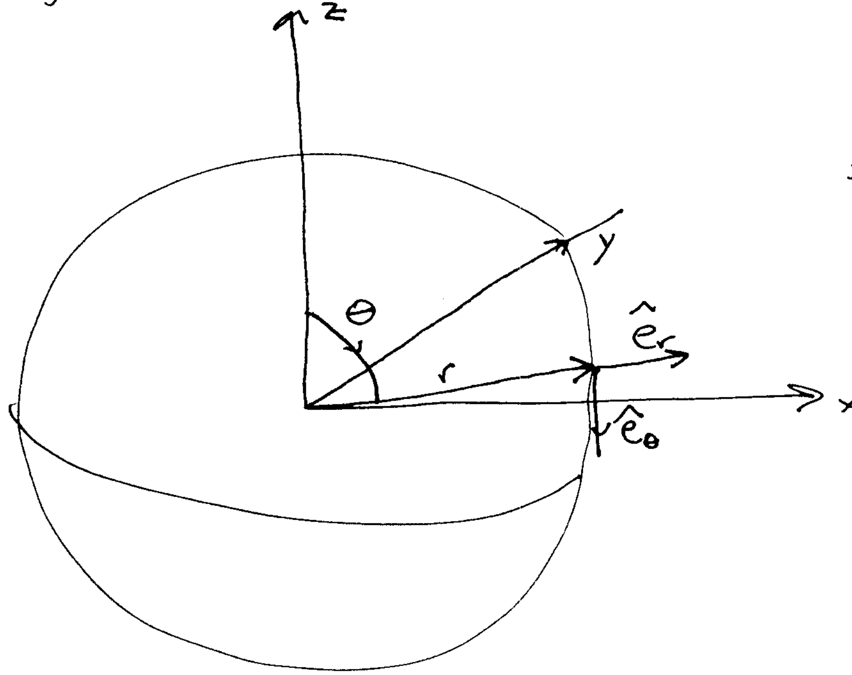
$$- R\dot{\phi}^2\hat{e}_r + \ddot{z}\,\hat{e}_z$$

$$\boxed{\vec{a}(t) = (\ddot{R} - R\dot{\phi}^2)\,\hat{e}_r + (2\dot{\phi}\dot{R} + R\ddot{\phi})\,\hat{e}_\phi + \ddot{z}\,\hat{e}_z}$$

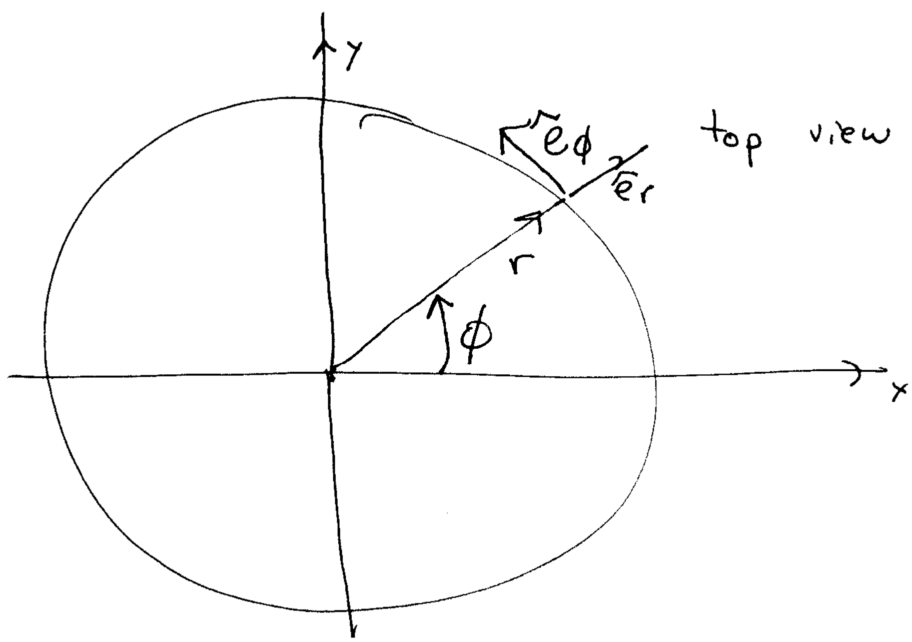
$$+ R\ddot{\phi}$$

Section - 3.4 - Spherical Coordinates

We can repeat the analysis for spherical coordinates, the following results.



side view



top view

Triple - $e_r \; e_\theta \; e_\phi$

Example

Problem In spherical coordinates,

$$\hat{e}_r \times \hat{e}_\theta = \hat{e}_\phi$$

Find $\hat{e}_r \times \hat{e}_\phi$ using BAC CAB

Sln

$$\hat{e}_r \times (\hat{e}_r \times \hat{e}_\theta) = \hat{e}_r \times \hat{e}_\phi$$

→ Note – Non-associative

$$= \overset{B}{\hat{e}_r}(\overset{A}{\hat{e}_r} \cdot \overset{C}{\hat{e}_\theta}) - \overset{C}{\hat{e}_\theta}(\overset{A}{\hat{e}_r} \cdot \overset{B}{\hat{e}_r})$$

$$= 0 - \hat{e}_\theta$$

$$\hat{e}_r = \hat{\imath}\sin\theta\cos\phi + \hat{\jmath}\sin\theta\sin\phi + \hat{k}\cos\theta$$

$$\hat{e}_\theta = \hat{\imath}\cos\theta\cos\phi + \hat{\jmath}\cos\theta\sin\phi - \hat{k}\sin\theta$$

$$\hat{e}_\phi = -\hat{\imath}\sin\phi + \hat{\jmath}\cos\phi$$

Position

$$\vec{r}(t) = r(t)\,\hat{e}_r$$

Velocity

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \dot{r}\,\hat{e}_r + r\dot{\phi}\sin\theta\,\hat{e}_\phi + r\dot{\theta}\,\hat{e}_\theta$$

Acceleration

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\,\hat{e}_r$$
$$+ (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\,\hat{e}_\theta$$
$$+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\,\hat{e}_\phi$$

Section 3.6 - Example

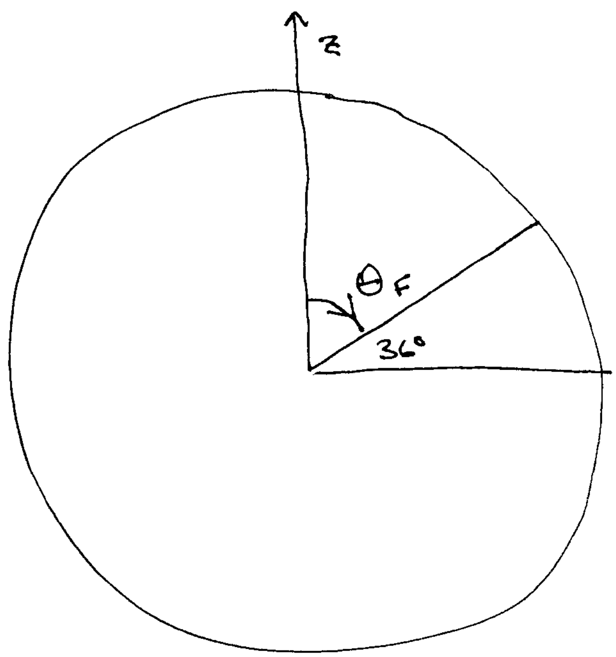
Problem - On our trip to Texas over Christmas, we drove South about 250 miles at an average speed of 70 mph, the West 100 miles. Compute the acceleration in a coordinate system with origin at center of earth in which the earth rotates once a day.

Sln - Look up constants online -

Radius of Earth

$$R_{earth} = 6.37 \times 10^{6}\ m$$

Location of Fayetteville (From Grenwich) $36^\circ N$ $94^\circ W$



$$\theta_F = \left(\frac{90^\circ - 36^\circ}{360^\circ}\right) 2\pi = 0.94\ rad$$

Choose $\phi_F = 0$ at beginning of trip, will not affect acceleration.

Compute Angular Velocity of Earth (ω_e) - Radian per second
about z-axis

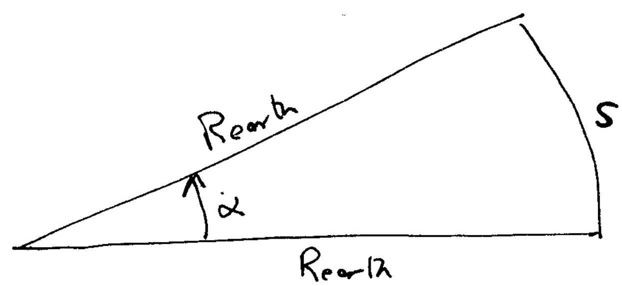
$$\omega_e = \frac{2\pi}{\text{day}} = 7.2722 \times 10^{-5} \text{ rad/s}$$

(actually 7.29×10^{-5})

Compute Angular Velocity of Car -

$$\text{Speed } v = (70 \text{ mph})\left(\frac{1 \text{ hr}}{3600 \text{s}}\right)\left(\frac{1.609 \text{ km}}{\text{mi}}\right)\left(\frac{1000 \text{ m}}{1 \text{ km}}\right)$$

$$= 31.3 \text{ m/s}$$



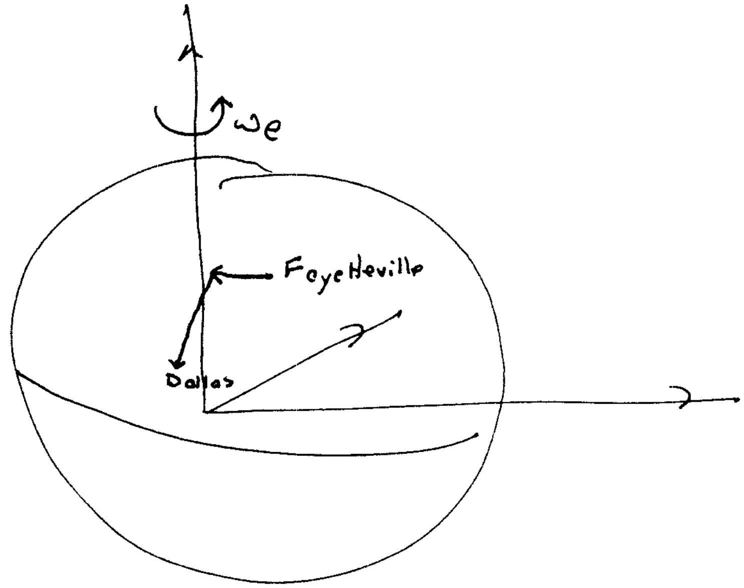
$$|\omega_c| = \dot{\alpha} = \frac{\dot{s}}{R_{earth}} = \frac{v}{R_{earth}}$$

$$= 4.9 \times 10^{-6} \text{ rad/s}$$

~~Write location in spherical coordinates~~

~~West~~

$$\cancel{r(t) = R_{earth}, \quad \theta(t) = \theta_F + \ldots, \quad \phi(t) = \omega_e t -}$$



Write location in spherical coordinates,

West

$$r(t) = R_{earth}, \quad \theta(t) = \theta_F, \quad \phi(t) = \omega_e t - \omega_c t$$

$$\dot{r} = \ddot{r} = 0 \qquad \dot{\theta} = \ddot{\theta} = 0 \qquad \dot{\phi} = \omega_e - \omega_c, \quad \ddot{\phi} = 0$$

South

$$r(t) = R_{\cancel{earth}}, \quad \theta(t) = \omega_c t + \theta_F, \quad \phi(t) = \omega_e t$$

$$\dot{r} = \ddot{r} = 0 \qquad \dot{\theta} = \omega_c, \ \ddot{\theta} = 0 \qquad \dot{\phi} = \omega_e, \ \ddot{\phi} = 0$$

Velocity West

$$\vec{v}_W(t) = \dot{r}\,\hat{e}_r + r\dot{\phi}\sin\theta\,\hat{e}_\phi + \hat{e}_\theta\, r\dot{\theta}$$

$$= R_e \sin\theta_F (\omega_e - \omega_c)\,\hat{e}_\phi$$

Velocity South

$$\vec{v}_s(t) = R_e \omega_e \sin\theta_F\,\hat{e}_\phi + R_e \omega_c\,\hat{e}_\theta$$

Acceleration West

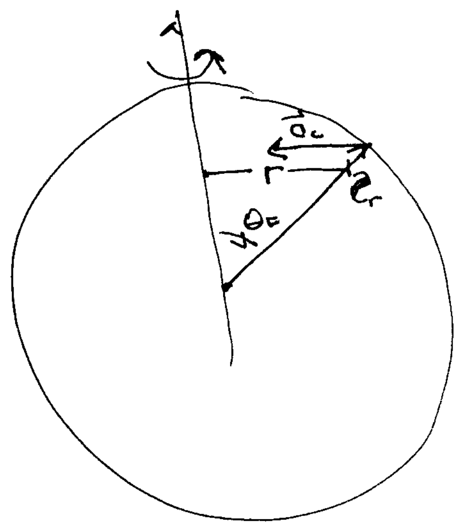
$$\vec{a}_W(t) = -R_e(\omega_e-\omega_c)^2\sin^2\theta_F\,\hat{e}_r$$

$$-R_e(\omega_e-\omega_c)^2\sin\theta_F\cos\theta_F\,\hat{e}_\theta$$

$$= -R_e(\omega_e-\omega_c)^2\left[\sin^2\theta_F\,\hat{e}_r + \sin\theta_F\cos\theta_F\,\hat{e}_\theta\right]$$

What? We know $a_c = \dfrac{v^2}{r} = r\omega^2$

if velocity constant.



So the acceleration is directed inward z-axis and the correct distance $r_{axis} = R_e\sin\theta_F$.

So we should find $\vec{a}_W \perp \hat{k}$, check it.

We need $\cancel{\hat{k}\cdot\vec{e}_\theta}$

$$\hat{k}\cdot\hat{e}_r = \cos\theta$$

$$\hat{k}\cdot\hat{e}_\theta = -\sin\theta$$

$$\hat{K} \cdot \vec{a}_w(t) = -R_e(\omega_e - \omega_c)^2 \left[\sin^2\theta_F \, \hat{K}\cdot\hat{e}_r + \sin\theta_F\cos\theta_F \, \vec{e}_\theta\cdot\hat{K} \right]$$

$$= -R_e(\omega_e - \omega_c)^2 \left[\sin^2\theta_F \cos\theta_F - \sin^2\theta_F \cos\theta_F \right]$$

$$= 0$$

$$\cancel{\vec{a}(t) = -R_{earth}(\omega_e - \omega_c)^2\left(\sin^2\theta\, e_r + \sin\theta\cos\theta\,\hat{e}_\theta\right)}$$

$$\cancel{\hat{k}\cdot\vec{a}(t) = -R_{earth}(\omega_e-\omega_c)^2\left(\cos\theta\sin^2\theta - \sin^2\theta\cos\theta\right) = 0}$$

$$\vec{a} \perp \hat{k}$$

Look in the x-z plane $\phi = 0$

$$\hat{\imath}\cdot\hat{e}_r = \sin\theta\cos\phi = \sin\theta$$

$$\hat{\imath}\cdot\hat{e}_\theta = \cos\theta\cos\phi = \cos\theta$$

$$\hat{\imath}\cdot\vec{a}(t) = -R_{earth}(\omega_e-\omega_c)^2\underbrace{\left[\sin^3\theta - \sin\theta\cos^2\theta\right]}_{\sin\theta\,\underbrace{\left[\sin^2\theta + \cos^2\theta\right]}_{1}}$$

$$\hat{\imath}\cdot\vec{a}(t) = -\underbrace{R_{earth}\sin\theta}_{\text{radial distance}}(\omega_e-\omega_c)^2$$

Section 3.5 – Good Stuff From Examples

(1) $\vec{v} \cdot \vec{a} = v\dot{v} \quad \Rightarrow \quad \vec{v} \perp \vec{a}$ only if $\dot{v} = 0$

(2) Tangential Acceleration – $a_T = \dfrac{\vec{v} \cdot \vec{a}}{v}$

Acceleration in direction of velocity

$\Rightarrow \quad a_T = \dot{v}$, time derivative of speed.

(3) <u>Normal Acceleration</u> $a_n = (a^2 - a_T^2)^{1/2}$

(4) <u>Angle Between Velocity and Acceleration</u> –

$$\cos\theta = \frac{\vec{v} \cdot \vec{a}}{va}$$