Lecture 6 - 1/27/03

Section 6.0 – Coverage

We ~~cover~~ skip sections – 3.5, 3.7, 3.8, 3.9

Section 6.1 - Simple harmonic oscillation

$$F = -kx$$

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \frac{k}{m}x = 0$$

$$\sqrt{\frac{k}{m}} = \omega_0$$

$$\ddot{x} + \omega_0^2 x = 0$$

<u>Guess a solution</u> Need 2 parameters to fix x_0 and v_0

$$x(t) = A\sin(\omega t + \phi)$$

<u>Recall</u> $\omega \equiv$ Angular Frequency

$f \equiv$ Frequency

$$2\pi f = \omega$$

$$\text{Period } T = \frac{1}{f} = \frac{2\pi}{\omega}$$

<u>Example</u> — A particle of mass m receives a push at $t=0$ giving it a velocity v_0 at $x_0=0$.

$$x(0) = A\sin\phi = 0 \Rightarrow \sin\phi = 0 \Rightarrow \phi = 0$$

$$\dot{x}(t) = A\omega\cos(\omega t+\phi)$$

$$\dot{x}(0) = v_0 = A\omega_0\cos(\phi) = A\omega_0$$

$$\boxed{A = \frac{v_0}{\omega_0}}$$

Section 6.2 - Energy Stored

$$\int_0^{2\pi} d\theta = 2\pi = \int_0^{2\pi} (\sin^2\theta + \cos^2\theta)\, d\theta$$

$$= 2\int_0^{2\pi} \sin^2\theta\, d\theta$$

$$\int_0^{2\pi} \sin^2\theta\, d\theta = \pi$$

Average Kinetic Energy of SHO

$$\langle K \rangle = \frac{1}{T_0}\int_0^{T_0} \frac{1}{2} m v^2\, dt$$

$$v = A\omega_0 \sin(\omega_0 t + \phi)$$

$$\langle K \rangle = \frac{m}{2T_0}\int_0^{T_0} A^2 \omega_0^2 \cos^2(\omega_0 t + \phi)\, dt$$

Change Variables $u = \omega_0 t$ $\quad du = \omega_0\, dt$

$$T_0 \omega_0 = T_0 2\pi f_0 = \frac{T_0 2\pi}{T_0} = 2\pi$$

$$\langle K \rangle = \frac{m A^2 \omega_0}{2T_0}\int_0^{2\pi} \cos^2(u + \phi)\, du$$

$$= \frac{\pi m A^2 \omega_0}{\cancel{2} T_0} = \frac{m A^2 \omega_0^2}{4}$$

Likewise $\langle V \rangle = \frac{1}{4} m\omega_0^2 A^2$

Total Energy

$$E = \langle K \rangle + \langle V \rangle = \frac{1}{2} m\omega_0^2 A^2 = \frac{1}{2} k A^2$$

Section 6.3 - Complex Numbers

$$\sqrt{-1} \equiv i$$

$$i^2 = -1$$

Complex # $\quad C \equiv \underset{\text{real}}{A + iB}$

Complex Conjugate $\quad C^* = A - iB$

$$C + C^* = 2A \qquad \text{Real}$$

<u>Complex Exponential</u>

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Section 6.4 - Other Solution SHO

$$\ddot{x} + \omega_0^2 x = 0$$

Used $\quad x(t) = A \sin(\omega_0 t + \phi_0)$

Parameter A, ϕ_0

Could also use

$$x(t) = A_1 \sin(\omega_0 t) + A_2 \cos(\omega_0 t)$$

Parameters A_1 A_2

<u>Use Complex Numbers,</u>

$$x(t) = \mathrm{Re}\left[A e^{i[\omega_0 t + \phi_0]}\right]$$

$$\ddot{x} = i^2 \omega_0^2 A e^{i(\omega_0 t + \phi_0)} = -\omega_0^2 A e^{i(\omega_0 t + \phi_0)}$$

$$+ \quad \omega_0^2 x \qquad \omega_0^2 A e^{i(\omega_0 t + \phi_0)}$$

$$0$$

$$x(t) = \mathrm{Re}\left[A e^{i\omega_0 t + \phi_0}\right] \qquad \text{solution}$$

$$= A \cos(\omega_0 t + \phi_0)$$

<u>Solve it formally</u>

Let $\quad D = \frac{d}{dt} \quad$ an operator

$$(D^2 + \omega_0^2) x = 0$$

<u>Factor</u> $(D + i\omega_0)(D - i\omega_0)\, x = 0$

$(D - i\omega_0)\, e^{i\omega_0 t} = 0$

$(D + i\omega_0)\, e^{-i\omega_0 t} = 0$

$$x(t) = A_1\, e^{i\omega_0 t} + A_2\, \bar{e}^{i\omega_0 t}$$

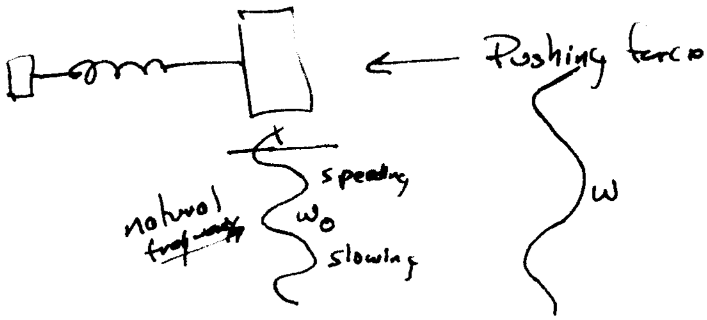
where A_1, A_2 complex

TOO Many Parameters, Let $A_2 = A_1^*$ since $x(t)$ real

$$x(t) = A_1\, e^{i\omega t} + A_1^*\, e^{-i\omega t}$$

TOO MUCH INFORMATION

6.5 - Driven Oscillators



Sometimes if pushing force not in tune, energy is removed instead of added. If force is in turn, energy is always added. In a SHO no energy is lost so BANG.

$$F_{ext} = F_0 \cos(\omega t)$$

EOM

$$\ddot{x} + \omega_0^2 x = \frac{F_{ext}}{m} = \frac{F_0}{m}\cos(\omega t)$$

$$= \mathrm{Re}\left[\frac{F_0}{m} e^{i\omega t}\right]$$

Propose Solution

$$x(t) = \mathrm{Re}\left[A e^{i(\omega t - \phi_0)}\right] \qquad A, \phi_0 \text{ real}$$

$$\ddot{x} = i^2 \omega^2 A e^{i\omega t - \phi_0} = -\omega^2 A e^{i(\omega t - \phi_0)}$$

$$-\omega^2 A e^{i(\omega t - \phi_0)} + \omega_0^2 A e^{i(\omega t - \phi_0)} = \frac{F}{m} e^{i\omega t}$$

$$(\omega_0^2 - \omega^2) A e^{-i\phi_0} = \frac{F_0}{m}$$

$$(\omega_0^2 - \omega^2) A = \frac{F_0}{m} e^{i\phi_0}$$

$$= \frac{F_0}{m}(\cos\phi_0 + i\sin\phi_0)$$

$\underline{Re}$ $\quad (\omega_0^2 - \omega^2) A = \frac{F_0}{m} \cos\phi_0$

Im $\quad 0 = i\sin\phi_0$

$\Rightarrow \phi_0 = 0$ No phase shift

$$A(\omega) = \frac{F_0/m}{\omega_0^2 - \omega^2}.$$

$$x_p(t) = A(\omega) \cos\omega t$$

what's p?! This is the particular solution. The complete solution is found by adding any solution to the homogeneous equation

$$\ddot{x}_h(t) + \omega_0^2 x_h(t) = 0$$

$$x(t) = \underbrace{x_h(t)}_{\text{transients}} + \underbrace{x_p(t)}_{\text{long term behavior}}$$

Lecture 7

Driven Oscillations

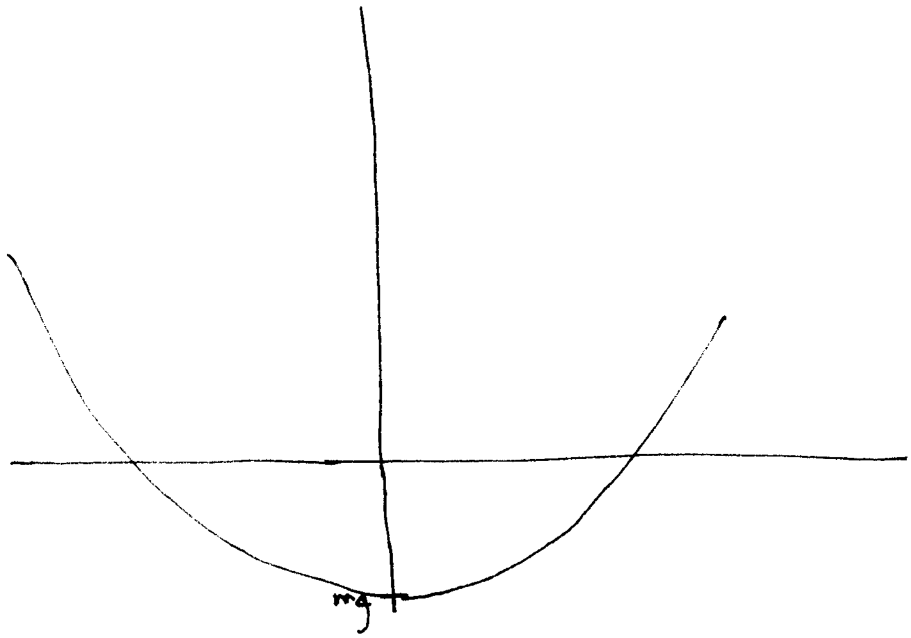
Section 7.0 - Admin

- Correct figure at top of 103
  $\omega_0 \rightarrow \omega_r$

- Class will be held in SCEN 110
  Friday, group homework session.

Section 7.1 Justin's Question

$$F = -mg + c_1 v + c_2 v^2 = 0$$



$$v^2 + \frac{c_1}{c_2} v - \frac{mg}{c_2} = 0$$

$$v_t = \frac{-\frac{c_1}{c_2} \pm \sqrt{\left(\frac{c_1}{c_2}\right)^2 + 4\frac{mg}{c_2}}}{2}$$

~~Pos~~ Other choice is

$$V_t = -\left[\frac{mg}{c_2} + \left(\frac{c_1}{2c_2}\right)^2\right] - \frac{c_1}{2c_2}$$

7.2 SHO

From our discussion of SHO

$$\ddot{x} + \omega_0^2 x = 0 \qquad \omega_0^2 = \frac{k}{m}$$

* $E = \frac{1}{2} k A^2 = \frac{1}{2} m \omega_0^2 A^2$

* where $x(t) = A \cos(\omega_0 t + \phi_0)$

Driven ~~by~~ $F_{ext} = F_0 \cos(\omega t)$

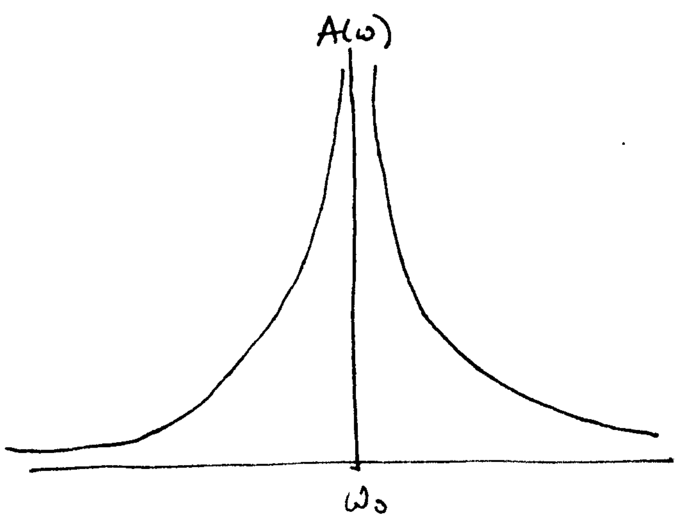
$$(-\omega^2 + \omega_0^2) A = \frac{F_0}{m} e^{i\phi} = \frac{F_0}{m}(\cos\phi_0 + i \sin\phi_0)$$

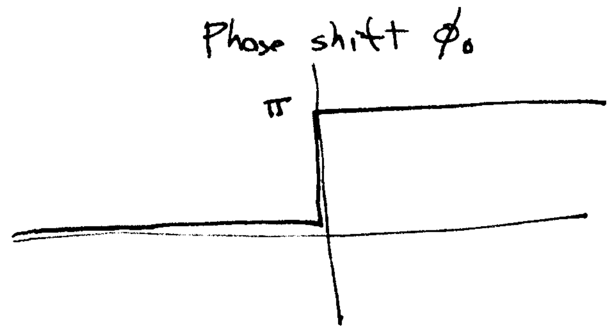
$$A(\omega) = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos\phi_0 \qquad > 0$$

$$\sin\phi_0 = 0 \qquad \Rightarrow \phi_0 = 0, \pi$$

for $\omega_0 > \omega$ $\quad \phi_0 = 0$

$\omega_0 < \omega$ $\quad \phi_0 = \pi$





7.3 Damped ~~Harmonic~~ Oscillator.

$$F = -kx - c\dot{x} = m\ddot{x}$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

$$2\gamma = \frac{c}{m} \qquad \omega_0^2 = \frac{k}{m}$$

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

$$\left(D^2 + 2\gamma D + \omega_0^2\right)x = 0$$

$$D = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2}$$

$$= -\gamma \pm \underbrace{\sqrt{\gamma^2 - \omega_0^2}}_{q}$$

q real — Overdamped, no oscillations $\gamma > \omega_0$

$q = 0$ — Critically damped $\gamma = \omega_0$

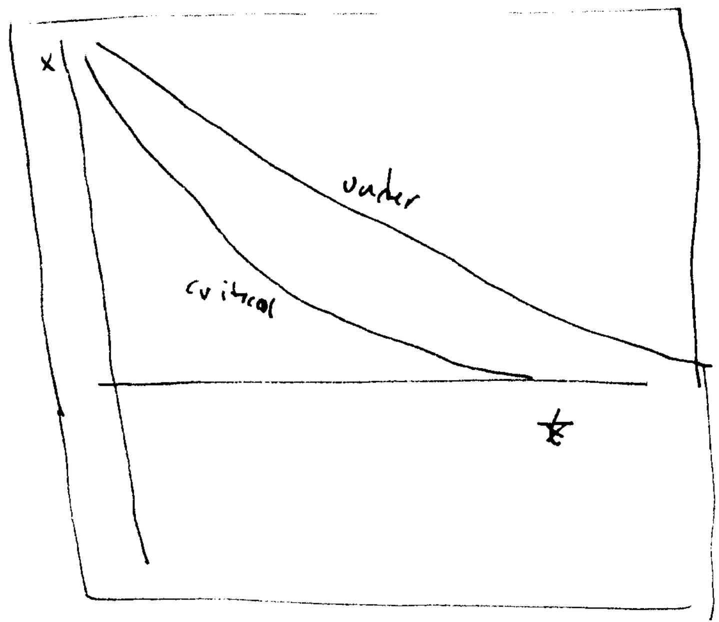
$\Rightarrow$ Fastest approach to equilibrium without oscillation.

q imaginary — Oscillations.

$$\left(D + (\gamma + q)\right)\left(D + (\gamma - q)\right)x = 0$$

$$x(t) = A_1 e^{-(\gamma+q)t} + A_2 e^{-(\gamma-q)t} \quad \text{Overdamped}$$

$$x(t) = A_1 e^{-\gamma t} + A_2 t e^{-\gamma t} \quad \text{Critically Damped}$$


x
under
critical
k

Section 7.3 Under damped Oscillations

q imaginary $\quad q = \sqrt{\gamma^2 - \omega_0^2} \quad \gamma < \omega_0$

$$= \sqrt{-1}\,\sqrt{\omega_0^2 - \gamma^2}$$

The frequency of damped oscillations will be

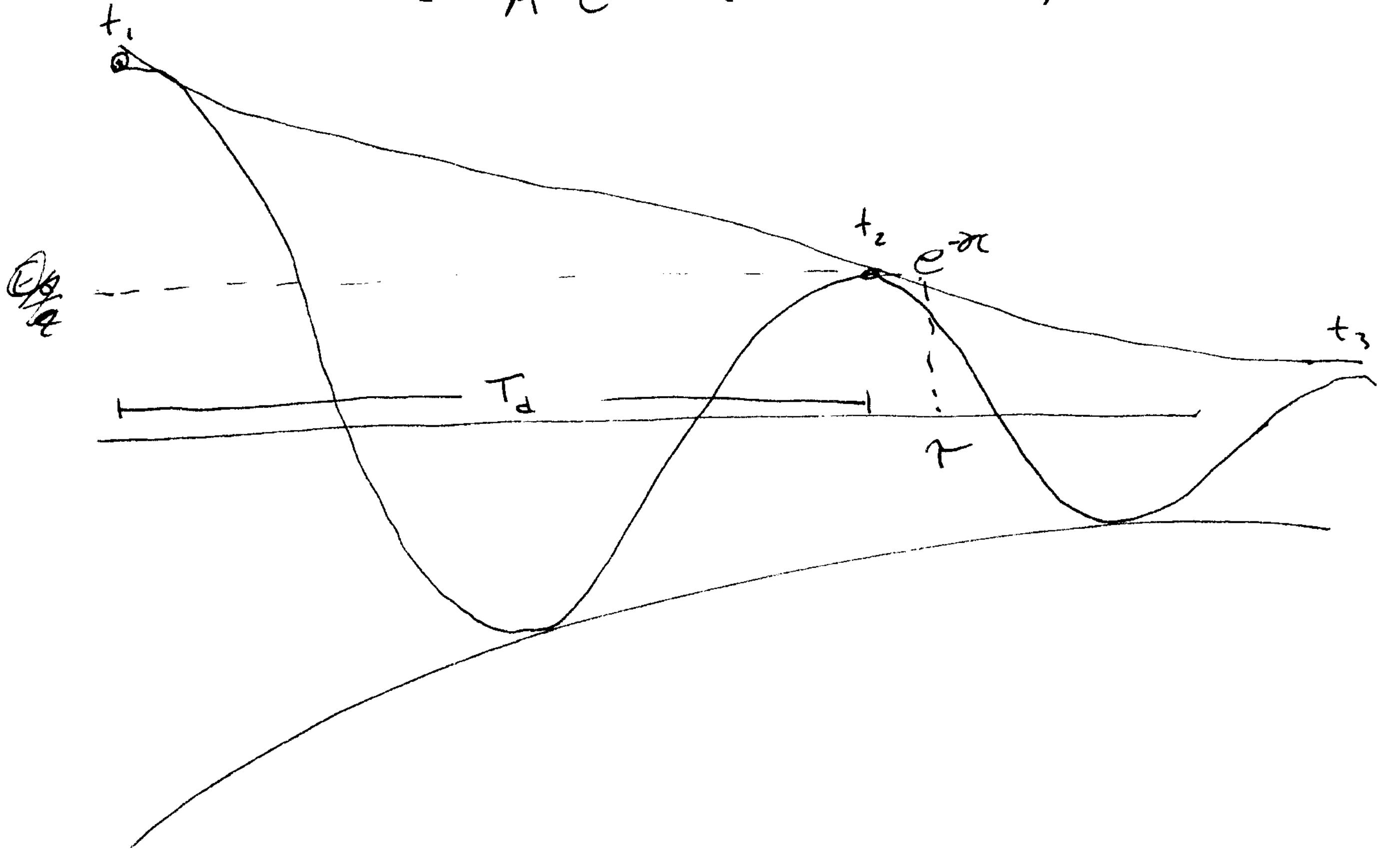
$$\omega_d = \sqrt{\omega_0^2 - \gamma^2}$$

$$q = i\omega_d$$

Solutions

$$x(t) = A_1 e^{-(\gamma + i\omega_d)t} + A_2^{T} e^{-(\gamma - i\omega_d)t}$$

$$= A e^{-\gamma t} \cos(\omega_d t + \phi_0)$$



Period of Damped Oscillations $T_d = \dfrac{2\pi}{\omega_d}$

Decay of Amplitude $A(t) = A e^{-\gamma t}$

Ratio of Amplitude of Successive Maxima

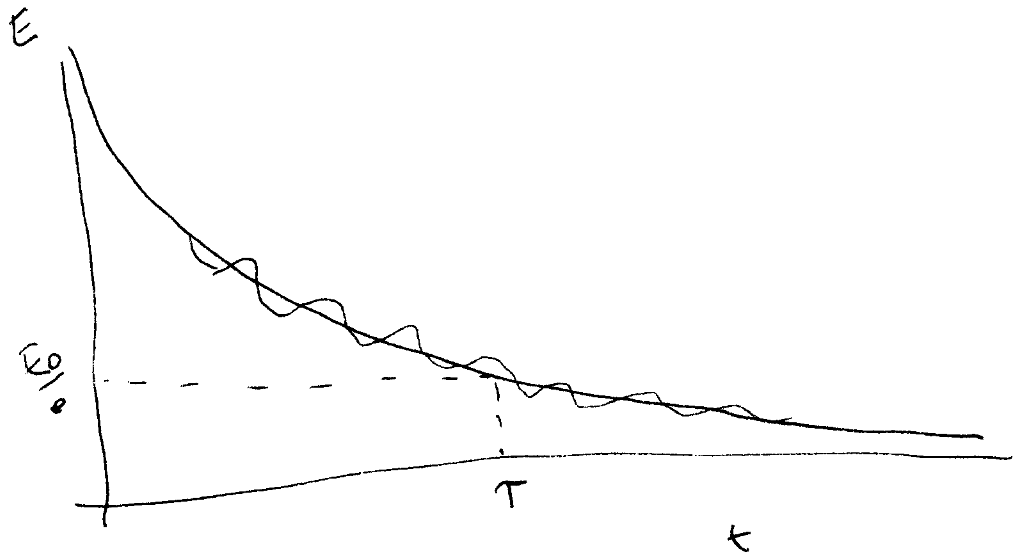
$$\frac{A(t_{n+1})}{A(t_n)} = e^{-\gamma T_d}$$

Decay of Energy – $E(t) = E_0 e^{-2\gamma t}$

since $E = \frac{1}{2} m \omega_0^2 A^2$

Characteristic Decay Time – $\tau = \dfrac{1}{2\gamma}$

$$E(t) = E_0 e^{-t/\tau}$$



$$\text{Quality} \equiv \frac{2\pi \cdot \text{Energy Stored}}{\text{Energy Lost one Cycle}}$$

$$Q = \frac{2\pi}{(T_d/\tau)} = \omega_d \tau = \frac{\omega_d}{2\gamma}$$

7.4 Driven Oscillations

Add Driving Force $\qquad F = F_0 \cos \omega t$

ω → Driving Frequency

EOM

$$\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$$

$$= \mathrm{Re}\left[\frac{F_0}{m} e^{i\omega t}\right]$$

Propose $\quad x(t) = A e^{i(\omega t - \phi_0)}$

$$\frac{d^2}{dt^2}\left(A e^{i(\omega t - \phi_0)}\right) + 2\gamma \frac{d}{dt} A e^{i(\omega t - \phi_0)} + \omega_0^2 A e^{i(\omega t + \phi_0)}$$

$$= \frac{F}{m} e^{i\omega t}$$

$$\boxed{\text{Solve the complex EOM and make sure things end up real at the end.}}$$

$$-\omega^2 A e^{i(\omega t - \phi_0)} + 2\gamma i \omega A e^{i(\omega t - \phi_0)} + A\omega_0^2 e^{i(\omega t + \phi_0)}$$

$$= \frac{F_0}{m} e^{i\omega t}$$

$$-\omega^2 A + 2\gamma i A\omega + A\omega_0^2 = \frac{F_0}{m} e^{i\phi_0}$$

$$= \frac{F_0}{m}(\cos\phi_0 + i \sin\phi_0)$$

$\underline{Re}$ $\quad A(\omega_0^2 - \omega^2) = \frac{F_0}{m} \cos\phi_0$

$\underline{Im}$ $\quad 2\gamma A \omega = \frac{F_0}{m} \sin\phi_0$

$\sqrt{R_e^2 + I_m^2}$ $\quad A^2(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2 A^2 = \left(\frac{F_0}{m}\right)^2$

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}$$

$$\tan\phi_0 = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}$$

The resonant frequency give maximum amplitude

$$A_{max}(\omega_r) = \frac{F_0/m}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$$

where $\quad \omega_r = \sqrt{\omega_0^2 - 2\gamma^2}$

$\underline{Note}$ $\quad$ No resonance if $\omega_0^2 < 2\gamma^2$

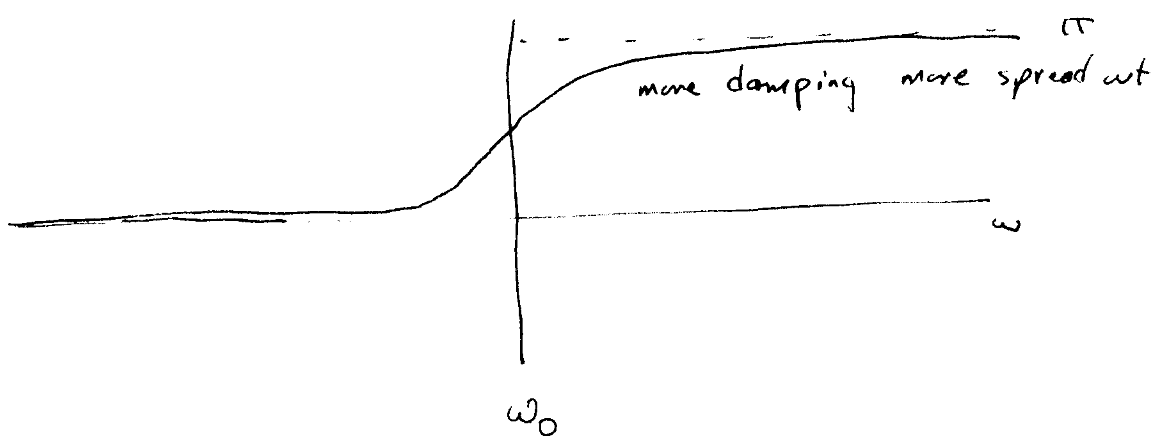
Quality - $Q = \frac{\omega_d}{2\gamma}$

Width at Half Maximum (Weak Damping) $\Delta\omega = 2\gamma$

$$\frac{\Delta\omega}{\omega_0} \approx \frac{1}{Q}$$

$\Rightarrow$ High Quality Narrow Resonance

Phase Shift



7.4(d)