Section - Overview of Lagrangians

Suppose we have a dynamical system composed of N mass and subject to M holonomic constraints.

Holonomic Constraints - $f(x_i, y_i, z_i, t) = 0$

$\Rightarrow$ Not velocity dependent

$\Rightarrow$ Example - Bead confined to hoop $x_i^2 + y_i^2 + z^2 = R^2$

$\Rightarrow$ Don't worry about terminology, almost any constraint that does not involve losses (friction) meets this criteria

Generalized Coordinates - Any set of coordinates that fully specifies the location of all mass in the system $\{q_i\}$

$\Rightarrow$ Problem with Newton's Laws, we have to use cartesian or other mutually orthogonal coordinates that are not the most convienient for specifying the system.

$\Rightarrow$ We have to deal with forces of explicitly.

Alternate Formulations of ~~New~~ Mechanics (all equivalent)

(1) Newton's Laws

(2) Hamilton's Variational Principle

(3) Hamilton's Equations

(4) D'Alembert's Principle (Virtual Work)

(5) Lagrange's Eqns.

Lagrange's Eqns - Weaknesses - Not intuitive

Strengths - Use any coordinate system you want. Forces of contraints taken care of automatically.

$\Rightarrow$ You want intuition use Newton, you want to to do a nasty calculation easily use Lagrange.

Lagragian - $L \equiv T - V$

Kinetic energy potential energy

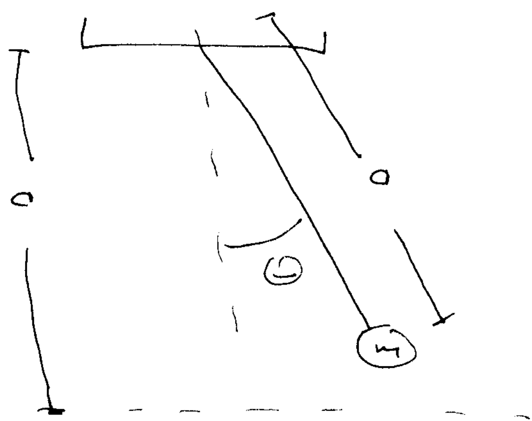
Lagrange's Equations - For each generalized coordinate q_i,

$$\frac{\partial L}{\partial q_i} - \frac{d}{dT}\frac{\partial L}{\partial \dot{q}_i} = 0$$

$\Rightarrow$ Equivalent to $\vec{F} = m\vec{a}$

Simple Examples

Simple Pendulum



$$T = \frac{1}{2} m (a\dot{\theta})^2$$

$$V = mg(a - a\sin\theta)$$

$$\Longrightarrow \boxed{\text{Generalized Coordinate } \theta = q_1}$$

$$L = T - V$$

<u>Lagragian</u>

$$= \frac{1}{2} m a^2 \dot{\theta}^2 - mga + mga\cos\theta$$

$$\frac{\partial L}{\partial q_1} = \frac{\partial L}{\partial \theta} = -mga\sin\theta$$

$$\frac{\partial L}{\partial \dot{q}_1} = \frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ma^2\ddot{\theta}$$

Apply Lagrange's Eqn

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$-mga\sin\theta - ma^2\ddot{\theta} = 0$$

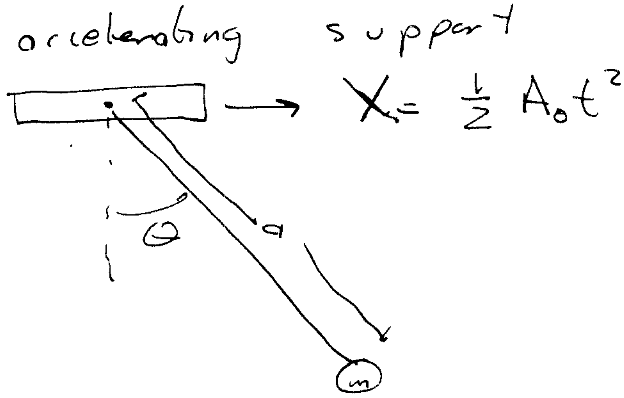
$$\boxed{\ddot{\theta} + \frac{g}{a}\sin\theta = 0}$$

⟹ Degree of Freedoms - Number of generalized coordinates required to full specify system $3N - M$

Well that wasn't easier. Newton worked better for simple system.

Section – A little harder

Simple pendulum with accelerating support



$\Rightarrow$ Moving constraint

$\Rightarrow$ General procedure for Lagrange

(I). Write location of all masses in terms of generalized coordinates.

(II). Compute kinetic energy and potential energy.

III Apply Lagrange

(I) Write locations in terms of generalized coordinates.

$$x = X + a \sin\theta$$

$$y = a - a\cos\theta$$

$$\dot{x} = \dot{X} + a\cos\theta\,\dot{\theta} = A_0 t + a\cos\theta\,\dot{\theta}$$

$$\dot{y} = a\sin\theta\,\dot{\theta}$$

II Compute kinetic energy

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right)$$

$$= \frac{1}{2} m \left(\left(A_0 t + a \cos\theta \dot{\theta} \right)^2 + a^2 \sin^2\theta \dot{\theta}^2 \right)$$

$$= \frac{1}{2} m A_0^2 t^2 + m A_0 t a \cos\theta \dot{\theta} + \frac{1}{2} m a^2 \dot{\theta}^2$$

$$V = mga(1 - \cos\theta)$$

$$L = T - V = \frac{1}{2} m A_0^2 t^2 + m A_0 a t \cos\theta \dot{\theta} + \frac{1}{2} m a^2 \dot{\theta}^2 - mga(1 - \cos\theta)$$

$$\frac{\partial L}{\partial \theta} = -m A_0 a t \sin\theta \dot{\theta} - mga \sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m A_0 a t \cos\theta + \cancel{2} m a^2 \dot{\theta}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = m A_0 a \cos\theta - m A_0 a t \sin\theta \dot{\theta} + m a^2 \ddot{\theta}$$

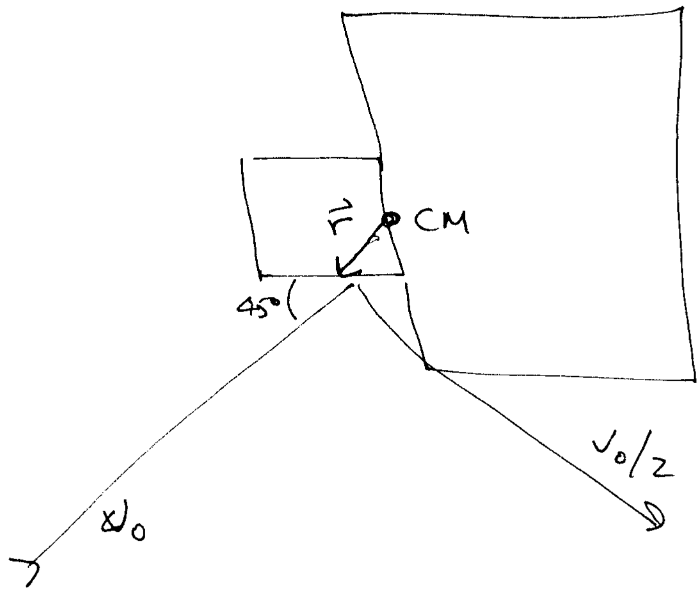
<u>Lagrange's Eqns</u>

$$-m A_0 a t \sin\theta \dot{\theta} - mga \sin\theta - m A_0 a \cos\theta + m A_0 a t \sin\theta \dot{\theta} - m a^2 \ddot{\theta} = 0$$

$$ma^2\ddot{\theta} + mga\sin\theta + mA_0 a\cos\theta = 0$$

sin

Section - Test Problem



Strategy - Linear and angular momentum conserved.
Energy?

$$\vec{P}_{truck} = \Delta \vec{p}_{ball}$$

$$\vec{v}_{CM, truck} = \frac{\vec{P}_{truck}}{M_{truck}}$$

$$\vec{L}_{truck} = \vec{r} \times \Delta \vec{p}_{truck} = I \omega$$

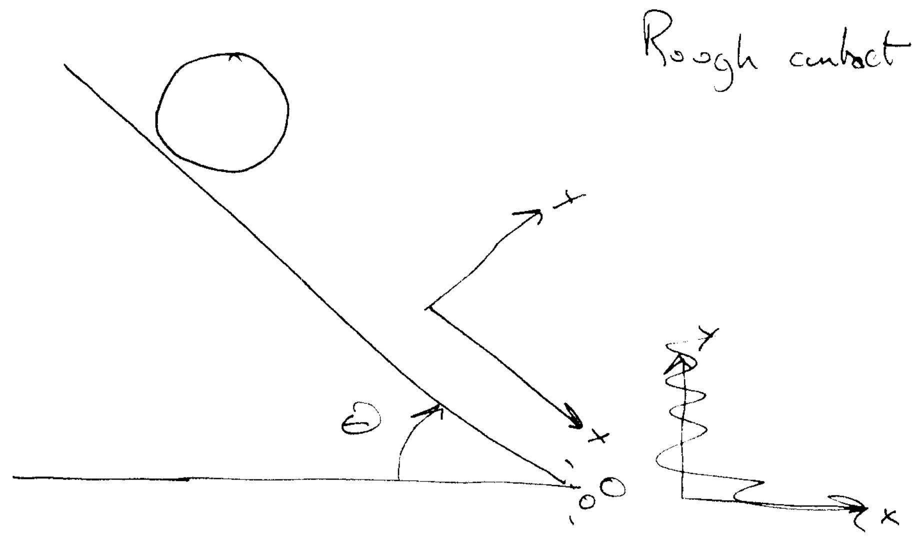
$$\omega_{truck} = \frac{L}{I}$$

$$I = I_1 + M_1\left(\frac{l_1}{2}\right)^2 + I_2 + M_2\left(\frac{l_2}{2}\right)^2 \quad \text{(Parallel axis thm)}.$$

$$E_{before} = \frac{1}{2} m_{ball} v_0^2 = \frac{1}{2} m_{ball} \left(\frac{v_0}{2}\right)^2 + \frac{1}{2} I \omega^2 + Q$$

if $Q = 0$ elastic.

Section - Another Lagrange



$$V(x) = -mgx\sin\theta$$

condition of rolling.

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{I}{a^2}\dot{x}^2 + \frac{1}{2}m\dot{x}^2$$

$$L = \frac{1}{2}\left(m + \frac{I}{a^2}\right)\dot{x}^2 + mgx\sin\theta$$

$$\frac{\partial L}{\partial x} = +mg\sin\theta$$

$$\frac{\partial L}{\partial \dot{x}} = \left(m + \frac{I}{a^2}\right)\dot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \left(m + \frac{I}{a^2}\right)\ddot{x}$$

Lagrange's Eqn

$$\frac{\partial L}{\partial x} - \frac{d}{dT}\frac{\partial L}{\partial \dot{x}} = 0$$

$$+mg\sin\theta - \left(m + \frac{I}{d^2}\right)\ddot{x} = 0$$

$$\ddot{x} = \frac{mg\sin\theta}{m + I/a^2}$$

If $\quad I = 2/5\, ma^2 \quad$ (sphere)

$$\ddot{x} = \frac{5}{7} g \sin\theta$$

Lecture 4/23

- Do 10.18 instead of 10.17

- Work 10.19 down to eqns ~~of~~ (r, θ) then

  use $\ddot{\theta} = \dfrac{d\dot{\theta}}{dt} = \dfrac{d\theta}{dt}\dfrac{d\dot{\theta}}{d\theta} = \dot{\theta}\dfrac{d\dot{\theta}}{d\theta}$

  to integrate θ eqn.

- Do 10.27(a) only.

Section - Ignorable Coordinates

~~Suppose L is not~~

Generalized Momenta $p_i = \frac{\partial L}{\partial \dot{q}}$

$$\boxed{\text{why?} \quad L = T = \tfrac{1}{2} m \dot{x}^2 \qquad \frac{\partial L}{\partial \dot{x}} = m\dot{x} = p}$$

Ignorable Coordinate — Suppose L is not a function of q_i, then Lagrange's eqn for q_i is

$$\underbrace{\frac{\partial L}{\partial q_i}}_{0} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

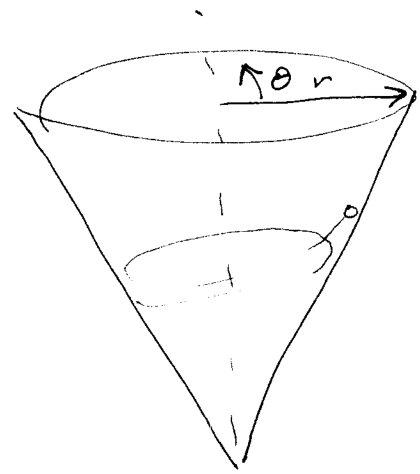
$$p_i = \text{constant.}$$

$\Longrightarrow$ The generalize momenta is conserved.

Section - Example with conserved momenta

Ex Motion of particle confined to cone,

surface of cone $z = \gamma r$ in polar.



What kind of motion do we expect?

Generalized coordinate r, θ

$$V(r,\theta) = mgz = mg\gamma r$$

$$\Rightarrow T = \frac{1}{2} m \vec{v} \cdot \vec{v}$$

$\Rightarrow$ Velocity in polar coordinates (recall)

$$\vec{v}(r,\theta) = \dot{r}\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + \dot{z}\hat{e}_z$$

$$T = \frac{1}{2} m\vec{v}\cdot\vec{v} = \frac{1}{2} m\left[\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right]$$

$$= \frac{1}{2} m\left[\dot{r}^2 + r^2\dot{\theta}^2 + \gamma^2\dot{r}^2\right]$$

$$L = T - V$$

$$= \frac{1}{2} m \left[(1+\gamma^2)\dot{r}^2 + r^2\dot{\theta}^2 \right] - mg\gamma r$$

(Note - L is independent of θ) θ ignorable.

$$\frac{\partial L}{\partial \theta} = 0 \qquad \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} = p_\theta \text{ (angular momentum)}$$

$$\frac{d}{dt} p_\theta = 0 \qquad p_\theta = \text{constant.}$$

$$\frac{\partial L}{\partial r} = -mg\gamma \qquad \frac{\partial L}{\partial \dot{r}} = m(1+\gamma^2)\dot{r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m(1+\gamma^2)\ddot{r}$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = -mg\gamma - m(1+\gamma^2)\ddot{r} = 0$$

$$\ddot{r} = \frac{g}{1+\gamma^2}$$

$$\frac{\partial L}{\partial r} = \cancel{2}\, m r \dot{\theta}^2 - m g \gamma$$

$$\frac{\partial L}{\partial \dot{r}} = m(1+\gamma^2)\dot{r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m(1+\gamma^2)\ddot{r}$$

<u>Lagrage's Eqn</u>

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$$

$$\cancel{2}\, m r \dot{\theta}^2 - m g \gamma - m(1+\gamma^2)\ddot{r} = 0$$

$$\ddot{r} - \frac{r\dot{\theta}^2}{1+\gamma^2} + \frac{g\gamma}{1+\gamma^2} = 0$$