Section 1 - Stability

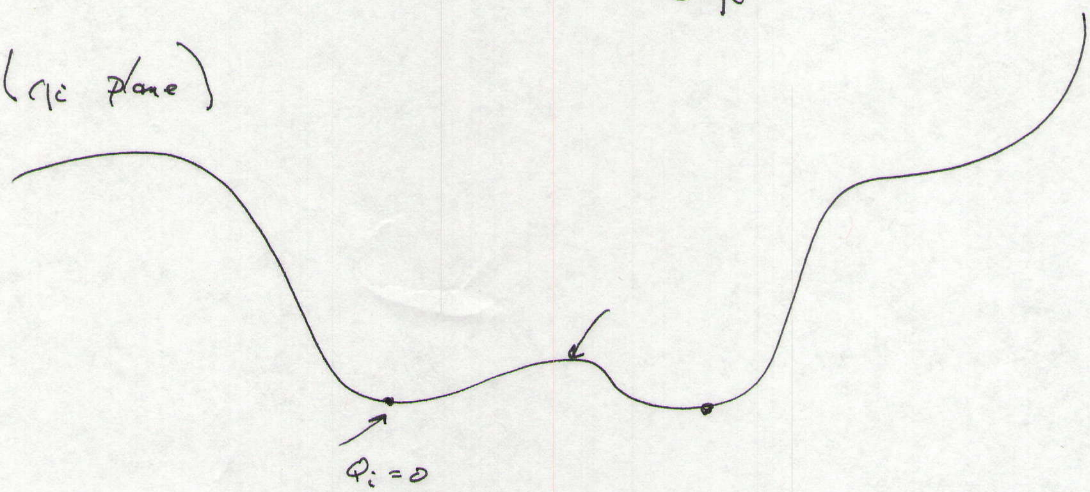
Ask Sloan
Textbook.

Suppose we have a potential function $V(q_1, q_2 \cdots q_n)$.

The generalized force ~~on th~~ on an object displaced by δq_i is $Q_i = -\dfrac{\partial V}{\partial q_i}$.

In equilibrium, $Q_i = 0 \implies \dfrac{\partial V}{\partial q_i} = 0$

V (q_i plane)



But, $\dfrac{\partial V}{\partial q_i} = 0$ does not produce stable equilibrium,

we also require $\dfrac{\partial^2 V}{\partial q_i^2} > 0$

But thats not really enough because the surface is complicated, we could expand $V(q_1 \cdots q_n)$ about the possible equilibrium

$$V \approx V_0 + \cancel{\sum_{ij} \frac{\partial^2 V}{\partial q_i \partial q_j} q_i q_j}$$

(Let $q_i = 0$ at

$$+ \frac{1}{2} (K_{11} q_1^2 + 2 K_{12} q_1 q_2 + 2 K_{13} q_1 q_3 + \cdots$$

$$K_{11} = \frac{\partial^2 V}{\partial q_1^2} \qquad K_{12} = \frac{\partial^2 V}{\partial q_1 \partial q_2}$$

Stable if $\begin{vmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{vmatrix} > 0$

Section 2 - Small Oscillations

Suppose we have a Lagrangian whose potential meets the condition of stable equilibrium, if the system is slightly disturbed from equilibrium it will oscillate about its equilibrium location $\vec{q}_0$.

Reducing Lagranging to Quadratic Form - If oscillations are small, we can expand the Lagrangian keep only to second-order. For example,

$$(\cos\theta)\dot{\theta} \longrightarrow \left(1 + \frac{\theta^2}{2} + \cdots\right)\dot{\theta}$$

$$\approx \dot{\theta} \quad \text{to second-order.}$$

Many times the Lagrangian will separate into a potential and kinetic quadratic form,

$$L = \underbrace{\frac{1}{2}\sum_{ij} M_{ij}\dot{q}_i\dot{q}_j}_{\text{kinetic}} - \underbrace{\frac{1}{2}\sum_{ij} K_{ij} q_i q_j}_{\text{potential}}$$

$$K_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}$$

Apply Lagrange's Eqn

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

$$\cancel{= \sum_j \frac{\partial L}{\partial q_i}} \quad - \sum_j \left(K_{ij} q_j + M_{ij} \ddot{q}_j \right) = 0$$

[Looks like ~~Harmonic~~ Oscillaters Right]

[Looks like matrix multiplication]

$$\overleftrightarrow{K}\vec{q} + \overleftrightarrow{M}\ddot{\vec{q}} = 0$$

Guess

constant

$$\vec{q} = \vec{a}\cos(\omega t + \delta)$$

$$\ddot{\vec{q}} = -\omega^2 \vec{a}\cos(\omega t + \delta)$$

$$\overleftrightarrow{K}\vec{q} - \omega^2 \overleftrightarrow{M}\vec{q} = 0$$

$$(\overleftrightarrow{K} - \omega^2 \overleftrightarrow{M})\vec{q} = 0 \qquad \text{Eigenvalue Problem}$$

<u>Normal Mode</u> - The solution we proposed $\vec{q} = \vec{a}\cos(\omega t + \sigma)$ has the entire system vibrating together with the same frequency.

$\Rightarrow$ In general, each mass vibrates with its own frequency.

$\Rightarrow$ Any vibration can be formed as a sum of normal modes

$$\vec{q}(t) = \sum A_i \vec{a}_i \cos(\omega_i t + \sigma)$$

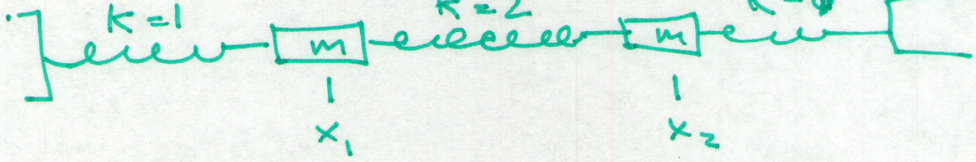
$\Rightarrow$ Normal mode frequencies ω_i and amplitudes $\vec{a}_i$ are eigenvalues and eigenvectors of $\overleftrightarrow{K} - \omega^2 \overleftrightarrow{M}$

$\Rightarrow$ If system is given amplitude $\vec{a}_i$, the system vibrates with only frequency ω_i.

Normal Frequencies ω_i

$$\det\left|\overleftrightarrow{K} - \omega^2 \overleftrightarrow{M}\right| = 0$$

Section 3 - Example of Normal Modes



$$L = \underbrace{\tfrac{1}{2} m\dot{x}_1^2 + \tfrac{1}{2} m\dot{x}_2^2}_{T} - \underbrace{\left[\tfrac{1}{2}(1)\,x_1^2 + \tfrac{1}{2}(1)\,x_2^2 + \tfrac{1}{2}(2)(x_2 - x_1)^2\right]}_{V}$$

$$= \tfrac{1}{2} m\dot{x}_1^2 + \tfrac{1}{2} m\dot{y}_2^2 - \tfrac{1}{2}\left[x_1^2 + x_2^2 + 2x_2^2 - 4x_1x_2 + 2x_1^2\right]$$

x_1 eqn

$$0 = -3x_1 + 2x_2 - m\ddot{x}_1$$

x_2 eqn

$$0 = -3x_2 + 2x_1 - m\ddot{x}_2$$

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Propose $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cos(\omega t + \delta)$ ↙ constants

$$\ddot{\vec{x}} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (-\omega^2) \cos(\omega t + \delta) = -\omega^2 \vec{x}$$

$$\begin{pmatrix} -m\omega^2 & 0 \\ 0 & -m\omega^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 3-m\omega^2 & -2 \\ -2 & 3-m\omega^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\det \begin{pmatrix} \; \end{pmatrix} = 0 = (3-m\omega^2)^2 - 4 = 0$$

$$9 - 6m\omega^2 + m^2\omega^4 - 4 = 0$$

$$m^2\omega^4 - 6m\omega^2 + 5 = 0$$

$\cancel{m\omega^2 = 1}$

$$m\omega^2 = \frac{6 \pm \sqrt{36 - 20}}{2}$$

$$m\omega^2 = \frac{6 \pm 4}{2} = 5, 1$$

$$\omega = \sqrt{\frac{5}{m}}, \quad \sqrt{\frac{1}{m}}$$

<u>Find Amplitudes</u>

$m\omega^2 = 1$

$$\begin{pmatrix} 3-1 & -2 \\ -2 & 3-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Let $x_1 = 1$, $\quad 2x_1 - 2x_2 = 0 \quad \Rightarrow x_2 = 1$

First normal mode (symmetric)

$$\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\left(\sqrt{\frac{1}{m}}\, t + \delta\right)$$

$m\omega^2 = 5$

$$\begin{pmatrix} 3-5 & -2 \\ -2 & 3-5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Let $x_1 = 1$

$$-2x_1 - 2x_2 = 0 \qquad x_2 = -1$$

$$\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\left(\sqrt{\frac{5}{m}}\, t + \delta \right)$$

<u>And vibration</u> $\Longrightarrow$ Superposition of normal modes

$$\vec{x} = A\vec{x}_1 + B\vec{x}_2$$

Section 2 - Small Oscillation

Suppose stability condition is met, then the masses m_j in the system which feel a force pushing them back toward equilibrium is they are slightly displaced.

If there are no losses, the masses will oscillate about equilibrium. The Lagrangian will be in general,

$$L = \underbrace{\frac{1}{2}\sum_{ij} M_{ij}\dot{q}_i\dot{q}_j}_{T} - \frac{1}{2}\sum_{ij} K_{ij}q_iq_j$$

$$K_{ij} = \frac{\partial^2 V}{\partial q_i q_j}$$

EOM

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

$$\Longrightarrow -\sum_j \left[K_{ij}q_j + M_{ij}\ddot{q}_j \right]$$

$$0 = \overleftrightarrow{M}\ddot{\vec{q}} + \overleftrightarrow{K}\vec{q}$$

Guess oscillating solution,

$$\vec{q} = \vec{a}\cos(\omega t + \sigma)$$

$$\ddot{\vec{q}} = -\omega^2 \vec{a}\cos(\omega t + \sigma) = -\omega^2 \vec{q}$$

$$\overleftrightarrow{K}\vec{q} = \omega^2 \overleftrightarrow{M}\vec{q} \qquad *$$

$\Longrightarrow$ Eigenvalue problem

<u>Normal Modes</u> - In general each mass will oscillate with more than one frequency, however if we establish the system initially in one of the eigenvectors of $*$ the system will oscillate collectively with one frequency ω the eigenvector

$\Longrightarrow$ The normal modes are a complete basis for the ~~problem.~~ system, so the state

$$\vec{q}_{t_0} = \sum_j A_j \vec{a}_j \cos(\omega_j t + \sigma_j)$$