Small Oscillations

Simple Harmonic Oscillator (SHO) –

Restoring force $F = -kx$

Newton $m\ddot{x} = -kx$

EOM $m\ddot{x} + kx = 0$

$$\ddot{x} + \omega^2 x = 0$$

Frequency $\omega = \sqrt{\frac{k}{m}}$

Solution $x(t) = A\cos\omega t + B\sin\omega t$

$\Rightarrow$ Second Order so two parameters

$\Rightarrow$ Initial Conditions

$$x(0) = x_0 \qquad \dot{x}(0) = \dot{x}_0$$

Lagrangian $L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

$$V = \frac{1}{2}kx^2$$

$$F = -\frac{dV}{dx} = -kx \quad \checkmark$$

EOM

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \qquad \text{Lagrange's Eqn}$$

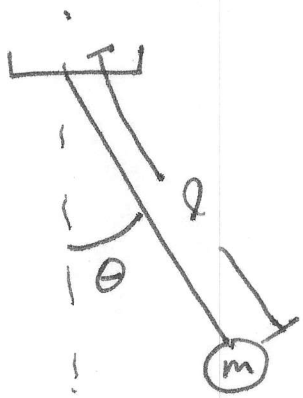
$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial x} = -kx$$

$$\boxed{m\ddot{x} + kx = 0}$$

Lots of things oscillate about an equilibrium point.

Very few are simple linear systems exactly.

Simple Pendulum



$$T = \frac{m}{2}(l\dot{\theta})^2 = \frac{m}{2}l^2\dot{\theta}^2 \qquad V = mgh = mg(l - l\cos\theta)$$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl + mgl\cos\theta$$

Not quadratic in $\theta, \dot{\theta}$ $\Rightarrow$ EOM will not be linear

$\Rightarrow$ No simple solutions.

Consider small oscillations — Keep terms to $\mathcal{O}(2)$

in $\theta, \dot{\theta}$ so terms like θ^2, $\theta\dot{\theta}$, $\dot{\theta}^2$

$$\cos\theta = 1 - \frac{1}{2}\theta^2$$

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 - \frac{1}{2} m g l \theta^2$$

EOM

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \qquad \frac{\partial L}{\partial \theta} = -mgl\theta$$

Lagrange

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$m l^2 \ddot{\theta} + m g l \theta = 0$$

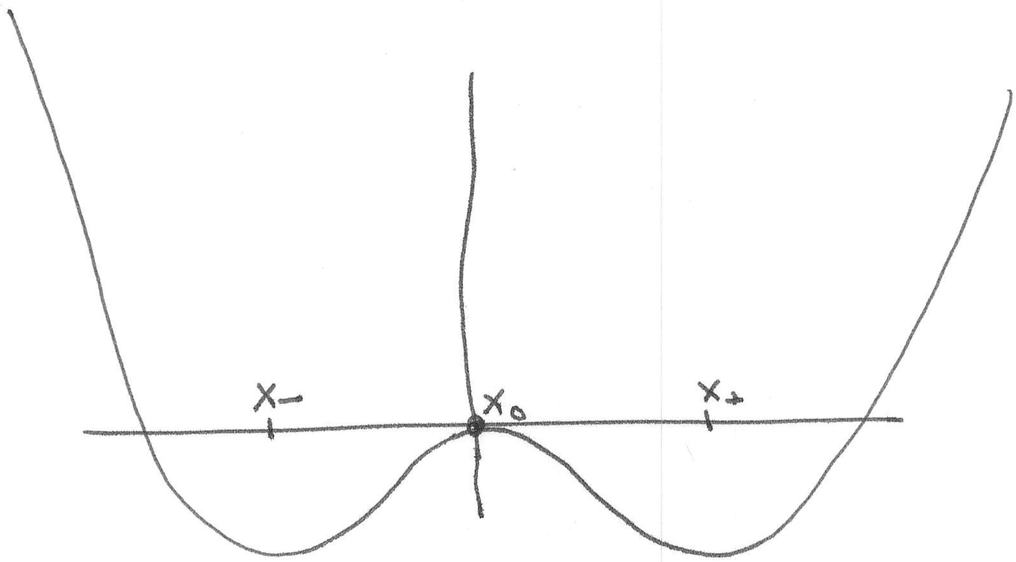
$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\omega = \sqrt{g/l}$$

But I cheated, $\theta = 0$ was accidentally the equilibrium point.

<u>Consider</u>

$$V(x) = ax^4 - bx^2$$



Should be in the news this year with the Higgs particle.

<u>Equilibrium points</u>

$$F = 0 \Rightarrow \frac{\partial V}{\partial x} = 0$$

$$\frac{\partial V}{\partial x} = 4ax^3 - 2bx = 0$$

$$x_0 = 0 \,, \quad 4ax^2 - 2b = 0 \Rightarrow x_\pm = \pm\sqrt{\frac{b}{2a}}$$

<u>Stability</u> How do we tell if an equilibrium point is stable? Energy curve must be a minimum

$$\frac{\partial^2 V}{\partial x^2} > 0$$

$$\frac{\partial^2 V}{\partial x^2} = 12ax^2 - 2b$$

For x_0, $\frac{\partial^2 V}{\partial x^2} = -2b < 0$ unstabe

$x_\pm$, $\frac{\partial^2 V}{\partial x^2} = 12a\left(\frac{b}{2a}\right) - 2b = 4b > 0$

stable.

Look for oscillations about x_+

$$L = \frac{1}{2}m\dot{x}^2 - V = \frac{1}{2}m\dot{x}^2 - ax^4 + bx^2$$

Linearize about x_+ (Taylor expand)

$$V(x) = V(x_+) + \left.\frac{\partial V}{\partial x}\right|_{x_+}(x-x_+) + \frac{1}{2}\left.\frac{\partial^2 V}{\partial x^2}\right|_{x_+}(x-x_+)^2 + \cdots$$

Ignore $V(x_+)$ because it can't affect the dynamics

$$\left.\frac{\partial V}{\partial x}\right|_{x_+} = 0$$ since equilibrium.

$$\frac{\partial^2 V}{\partial x^2} = 12ax^2 - 2b = 4b$$

To x^2,

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}(4b)(x-x_+)^2$$

Now, let $x_1 = x - x_+$

$$L = \frac{1}{2}m\dot{x}_1^2 - 2b\,x_1^2$$

$$\frac{\partial L}{\partial x_1} = -4bx_1 \qquad \frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0$$

$$m\ddot{x}_1 + 4bx_1 = 0$$

$$\ddot{x}_1 + \frac{4b}{m}x_1 = 0$$

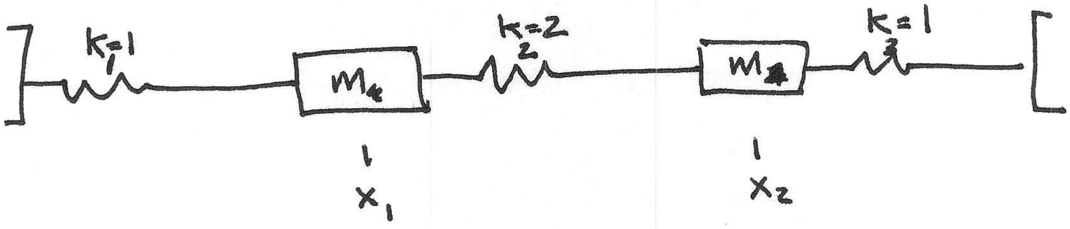
$$\omega^2 = \sqrt{\frac{4b}{m}}$$

Now, extend to more that one ~~particle~~ degree of freedom.

$$L = \cancel{T_1 + T_2 - V(} \quad T(x_1, x_2) - V(x_1, x_2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = 0$$

Ex



$$L = \underbrace{\tfrac{1}{2} m \dot{x}_1^2 + \tfrac{1}{2} m \dot{x}_2^2}_{T} - \underbrace{\left(\tfrac{1}{2} k_1 x_1^2 + \tfrac{1}{2} k_2 (x_2 - x_1)^2 + \tfrac{1}{2} k_3 x_2^2 \right)}_{V}$$

x_1 eqn

$$\frac{\partial L}{\partial x_1} = -k_1 x_1 + k_2 (x_2 - x_1)$$

$$\frac{\partial L}{\partial \dot{x}_1} = m \dot{x}_1$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0 \implies m \ddot{x}_1 + (k_2 + k_1) x_1 - k_2 x_2$$

x_2 eqn

$$\frac{\partial L}{\partial x_2} = -k_2(x_2 - x_1) - k_3 x_2$$

$$\frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2$$

EOM $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = 0 = m\ddot{x}_2 + (k_2+k_2)x_2 - k_2 x_1$$

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} k_2+k_1 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\ddot{\vec{x}} + \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}\vec{x} = 0$$

Propose Solution

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\cos(\omega t + \delta)$$

$$\ddot{\vec{x}} = -\omega^2 \vec{x}$$

$$\begin{pmatrix} -m\omega^2 & 0 \\ 0 & -m\omega^2 \end{pmatrix} \vec{x} + \begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix} \vec{x} = 0$$

$$\begin{pmatrix} k_1+k_2-m\omega^2 & -k_2 \\ -k_2 & k_2+k_3-m\omega^2 \end{pmatrix} \vec{x} = 0$$

Has solution if $\det(\,) = 0$ $\Longrightarrow$ Looks like eigenvalue problem.

$$\det(\,) = (k_1+k_2-m\omega^2)(k_2+k_3-m\omega^2) - k_2^2 = 0$$

Let $k_1+k_2 = k_{12}$ $\quad k_2+k_3 = k_{23}$

$$k_{12}k_{23} + m^2\omega^4 - m\omega^2(k_{12}+k_{23}) - k_2^2 = 0$$

$$\cancel{\omega^2 = \frac{(k_{12}+k_{23}) \pm \sqrt{(k_{12}+k_{23}) - 4m^2(k_{12}k_{23} - k_2^2)}}{2m^2}}$$

Problem Gives $\quad k_1 = k \quad k_2 = 2k \quad k_3 = k$

$$k_{12} = 3k \qquad k_{23} = 3k$$

$$(3k)^2 + m^2\omega^4 - m\omega^2(6k) - 4k^2 = 0$$

$$m^2\omega^4 - 6k\,m\omega^2 + 5k^2 = 0$$

$$m\omega^2 = \frac{6k \pm \sqrt{36k^2 - 4\cdot 5k^2}}{2}$$

$$= \frac{6k \pm \sqrt{16k^2}}{2}$$

$$= \frac{6k \pm 4k}{2}$$

$$\boxed{\cancel{m}\omega^2 = 1\,\frac{k}{\cancel{m}} \quad \text{or} \quad 5\frac{k}{m}}$$

Normal Mode Frequencies

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad\qquad \omega_2 = \sqrt{5}\sqrt{\frac{k}{m}}$$

Any oscillation can be written as a superposition of normal modes. To excite a pure normal mode, we have to prepare the particles correctly.

$$\begin{pmatrix} k_{12} - m\omega^2 & -k_2 \\ -k_2 & k_{23} - m\omega^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

For $\omega_1 = \omega_0$ $\qquad\qquad \omega_0 = \sqrt{\frac{k}{m}}$

$$\begin{pmatrix} 3k - k & -2k \\ -2k & 3k - k \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

Let $a_1 = 1 \Rightarrow a_2 = 1$

$$2a_1 - 2a_2 = 0$$

One normal model is

$$\vec{x}_1 = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_1 t + \delta_1)$$

The other mode is $\omega_1 = \sqrt{5}\,\omega_0$

$$\begin{pmatrix} 3k - 5k & -2k \\ -2k & 3k - 5k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -2k & -2k \\ -2k & -2k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

If $a_1 = 1$, $-2a_1 - 2a_2 = 0 \Rightarrow a_2 = -1$

The second mode is

$$\vec{x}_2 = A \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t + \delta)$$