gives directly

$$\mathbf{a} = (-b\omega_2^2 \sin^2 \theta - b\omega_1^2)\mathbf{e}_r - b\omega_2^2 \sin \theta \cos \theta \, \mathbf{e}_\theta + 2b\omega_1\omega_2 \cos \theta \, \mathbf{e}_\phi$$

The point at the top has coordinate $\theta = 0$, so at that point

$$\mathbf{a} = -b\omega_1^2\mathbf{e}_r + 2b\omega_1\omega_2\mathbf{e}_\phi$$

The first term on the right is the centripetal acceleration, and the last term is a transverse acceleration normal to the plane of the wheel.

PROBLEMS

1.1 Given the two vectors $\mathbf{A} = \mathbf{i} + \mathbf{j}$ and $\mathbf{B} = \mathbf{j} + \mathbf{k}$, find the following:

(a) $\mathbf{A} + \mathbf{B}$ and $|\mathbf{A} + \mathbf{B}|$

(b) $3\mathbf{A} - 2\mathbf{B}$

(c) $\mathbf{A} \cdot \mathbf{B}$

(d) $\mathbf{A} \times \mathbf{B}$ and $|\mathbf{A} \times \mathbf{B}|$

1.2 Given the three vectors $\mathbf{A} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{B} = \mathbf{i} + \mathbf{k}$, and $\mathbf{C} = 4\mathbf{j}$, find the following:

(a) $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C})$ and $(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C}$

(b) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$

(c) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$

1.3 Find the angle between the vectors $\mathbf{A} = a\mathbf{i} + 2a\mathbf{j}$ and $\mathbf{B} = a\mathbf{i} + 2a\mathbf{j} + 3a\mathbf{k}$. (*Note:* These two vectors define a face diagonal and a body diagonal of a rectangular block of sides a, $2a$, and $3a$.)

1.4 Consider a cube whose edges are each of unit length. One corner coincides with the origin of an $Oxyz$ Cartesian coordinate system. Three of the cube's edges extend from the origin along the positive direction of each coordinate axis. Find the vector that begins at the origin and extends

(a) along a major diagonal of the cube;

(b) along the diagonal of the lower face of the cube.

(c) Calling these vectors $\mathbf{A}$ and $\mathbf{B}$, find $\mathbf{C} = \mathbf{A} \times \mathbf{B}$.

(d) Find the angle between $\mathbf{A}$ and $\mathbf{B}$.

1.5 Assume that two vectors $\mathbf{A}$ and $\mathbf{B}$ are known. Let $\mathbf{C}$ be an unknown vector such that $\mathbf{A} \cdot \mathbf{C} = u$ is a known quantity and $\mathbf{A} \times \mathbf{C} = \mathbf{B}$. Find $\mathbf{C}$ in terms of $\mathbf{A}$, $\mathbf{B}$, u, and the magnitude of $\mathbf{A}$.

1.6 Given the time-varying vector

$$\mathbf{A} = \mathbf{i}\alpha t + \mathbf{j}\beta t^2 + \mathbf{k}\gamma t^3$$

where α, β, and γ are constants, find the first and second time derivatives $d\mathbf{A}/dt$ and $d^2\mathbf{A}/dt^2$.

1.7 For what value (or values) of q is the vector $\mathbf{A} = \mathbf{i}q + 3\mathbf{j} + \mathbf{k}$ perpendicular to the vector $\mathbf{B} = \mathbf{i}q - q\mathbf{j} + 2\mathbf{k}$?

1.8 Give an algebraic proof and a geometric proof of the following relations.

$$|\mathbf{A} + \mathbf{B}| \le |\mathbf{A}| + |\mathbf{B}|$$
$$|\mathbf{A} \cdot \mathbf{B}| \le |\mathbf{A}|\,|\mathbf{B}|$$

1.9 Prove the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.

1.10 Two vectors **A** and **B** represent concurrent sides of a parallelogram. Show that the area of the parallelogram is equal to $|\mathbf{A} \times \mathbf{B}|$.

1.11 Show that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is not equal to $\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C})$.

1.12 Three vectors **A, B,** and **C** represent three concurrent edges of a parallelepiped. Show that the volume of the parallelepiped is equal to $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$.

1.13 Verify the transformation matrix for a rotation about the z-axis through an angle ϕ followed by a rotation about the y'-axis through an angle θ, as given in Example 1.8.2.

1.14 Express the vector $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ in the primed triad $\mathbf{i}'\mathbf{j}'\mathbf{k}'$ in which the $x'y'$-axes are rotated about the z-axis (which coincides with the z'-axis) through an angle of 30°.

1.15 Consider two Cartesian coordinate systems $Oxyz$ and $Ox'y'z'$ that initially coincide. The $Ox'y'z'$ undergoes three successive counterclockwise 45° rotations about the following axes: first, about the fixed z-axis; second, about its own x'-axis (which has now been rotated); finally, about its own z'-axis (which has also been rotated). Find the components of a unit vector **X** in the $Oxyz$ coordinate system that points along the direction of the x'-axis in the rotated $Ox'y'z'$ system. (*Hint: It would be useful to find three transformation matrices that depict each of the above rotations. The resulting transformation matrix is simply their product.*)

1.16 A racing car moves on a circle of constant radius b. If the speed of the car varies with time t according to the equation $v = ct$, where c is a positive constant, show that the angle between the velocity vector and the acceleration vector is 45° at time $t = \sqrt{b/c}$. (*Hint: At this time the tangential and normal components of the acceleration are equal in magnitude.*)

1.17 A small ball is fastened to a long rubber band and twirled around in such a way that the ball moves in an elliptical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b \cos \omega t + \mathbf{j}2b \sin \omega t$$

where b and ω are constants. Find the speed of the ball as a function of t. In particular, find v at $t = 0$ and at $t = \pi/2\omega$, at which times the ball is, respectively, at its minimum and maximum distances from the origin.

1.18 A buzzing fly moves in a helical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t + \mathbf{k}ct^2$$

Show that the magnitude of the acceleration of the fly is constant, provided b, ω, and c are constant.

1.19 A bee goes out from its hive in a spiral path given in plane polar coordinates by

$$r = be^{kt} \qquad \theta = ct$$

where b, k, and c are positive constants. Show that the angle between the velocity vector and the acceleration vector remains constant as the bee moves outward. (*Hint: Find* $\mathbf{v} \cdot \mathbf{a}/va$.)

1.20 Work Problem 1.18 using cylindrical coordinates where $R = b$, $\phi = \omega t$, and $z = ct^2$.

1.21 The position of a particle as a function of time is given by

$$\mathbf{r}(t) = \mathbf{i}(1 - e^{-kt}) + \mathbf{j}e^{kt}$$

where k is a positive constant. Find the velocity and acceleration of the particle. Sketch its trajectory.

1.22 An ant crawls on the surface of a ball of radius b in such a manner that the ant's motion is given in spherical coordinates by the equations

$$r = b \qquad \phi = \omega t \qquad \theta = \frac{\pi}{2}\left[1 + \frac{1}{4}\cos(4\omega t)\right]$$

Find the speed of the ant as a function of the time t. What sort of path is represented by the above equations?

1.23 Prove that $\mathbf{v} \cdot \mathbf{a} = v\dot{v}$ and, hence, that for a moving particle $\mathbf{v}$ and $\mathbf{a}$ are perpendicular to each other if the speed v is constant. (*Hint: Differentiate both sides of the equation* $\mathbf{v} \cdot \mathbf{v} = v^2$ *with respect to* t. *Note,* $\dot{v}$ *is not the same as* $|\mathbf{a}|$. *It is the magnitude of the acceleration of the particle along its instantaneous direction of motion.*)

1.24 Prove that

$$\frac{d}{dt}[\mathbf{r} \cdot (\mathbf{v} \times \mathbf{a})] = \mathbf{r} \cdot (\mathbf{v} \times \dot{\mathbf{a}})$$

1.25 Show that the tangential component of the acceleration of a moving particle is given by the expression

$$a_\tau = \frac{\mathbf{v} \cdot \mathbf{a}}{v}$$

and the normal component is therefore

$$a_n = (a^2 - a_\tau^2)^{1/2} = \left[a^2 - \frac{(\mathbf{v} \cdot \mathbf{a})^2}{v^2}\right]^{1/2}$$

1.26 Use the above result to find the tangential and normal components of the acceleration as functions of time in Problems 1.18 and 1.19.

1.27 Prove that $|\mathbf{v} \times \mathbf{a}| = v^3/\rho$, where ρ is the radius of curvature of the path of a moving particle.

1.28 A wheel of radius b rolls along the ground with constant forward acceleration a_0. Show that, at any given instant, the magnitude of the acceleration of any point on the wheel is $(a_0^2 + v^4/b^2)^{1/2}$ relative to the center of the wheel and is also $a_0[2 + 2\cos\theta + v^4/a_0^2b^2 - (2v^2/a_0b)\sin\theta]^{1/2}$ relative to the ground. Here v is the instantaneous forward speed, and θ defines the location of the point on the wheel, measured forward from the highest point. Which point has the greatest acceleration relative to the ground?

Problem 1.5 — $\vec{A}$, $\vec{B}$ known. $\vec{A} \cdot \vec{C} = \upsilon$ where υ is known. $\vec{A} \times C = \vec{B}$

Sln

$$\vec{B} \times (\vec{A} \times \vec{C}) = \vec{B} \times \vec{B} = 0$$

$$= (\vec{B} \times \vec{A}) \times \vec{C} = -(\vec{A} \times \vec{B}) \times \vec{C}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = 0$$

Using CAB rule

$$\vec{A} \times (\vec{B} \times \vec{C}) = 0 = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$$\vec{C} = \frac{\vec{B}(\vec{A} \cdot \vec{C})}{\vec{A} \cdot \vec{B}} = \frac{\upsilon \vec{B}}{\vec{A} \cdot \vec{B}}$$

Problem 1.6

$$A = \hat{i}\alpha t + \hat{j}\beta t^2 + \hat{k}\gamma t^3$$

$$\frac{dA}{dt} = \hat{i}\alpha + \hat{j}(2\beta t) + \hat{k}(3\gamma t^2)$$

$$\frac{dA^2}{dt} = \hat{j}(2\beta) + \hat{k}(6\gamma t)$$

Problem 1.7

$$\vec{A} = \hat{i}q + 3\hat{j} + \hat{k}$$

$$\vec{B} = \hat{i}q - q\hat{j} + 2\hat{k}$$

$\vec{A} \perp \vec{B}$ if $\vec{A} \cdot \vec{B} = 0$

$$\vec{A} \cdot \vec{B} = q^2 - 3q + 2 = 0$$

$$(q-1)(q-2) = 0$$

$q = 1$, $q = 2$ give $\vec{A} \perp \vec{B}$

Problem 1.8

Algebraically

$$|\vec{A}+\vec{B}|^2 = (\vec{A}+\vec{B})\cdot(\vec{A}\cdot\vec{B})$$

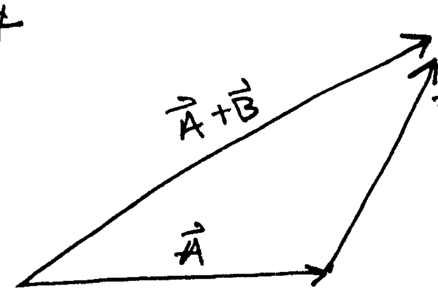
$$= A^2 + B^2 + 2\vec{A}\cdot\vec{B}$$

$$(|A|+|B|)^2 = |A|^2 + |B|^2 + 2|\vec{A}||\vec{B}|$$

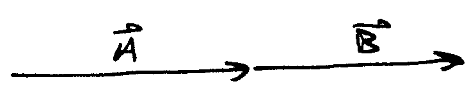
So $|\vec{A}+\vec{B}| \leq |\vec{A}|+|\vec{B}|$

because $2\vec{A}\cdot\vec{B} = 2|\vec{A}||\vec{B}|\cos\theta \leq 2|\vec{A}||\vec{B}|$

Geometrically



The length $\vec{A}+\vec{B}$ is at most $|\vec{A}|+|\vec{B}|$ because the longest $\vec{A}+\vec{B}$ can be is



and the length of the combination is $|\vec{A}|+|\vec{B}|$

Algebraically

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta \leq |\vec{A}||\vec{B}|$$

because $\cos\theta \leq 1$

Geometrically

Prove $|A \cdot \hat{B}| \leq |\vec{A}|$

The dot product projects $\vec{A}$ in the direction $\hat{n}$.
The projection cannot be longer than $\vec{A}$ itself.

Problem 1.9

$$A \times (\vec{B} \times \vec{C}) =$$

$$\vec{B} \times \vec{C} = \begin{pmatrix} i & j & k \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{pmatrix}$$

$$= (B_y C_z - B_z C_y)\,\hat{i}$$

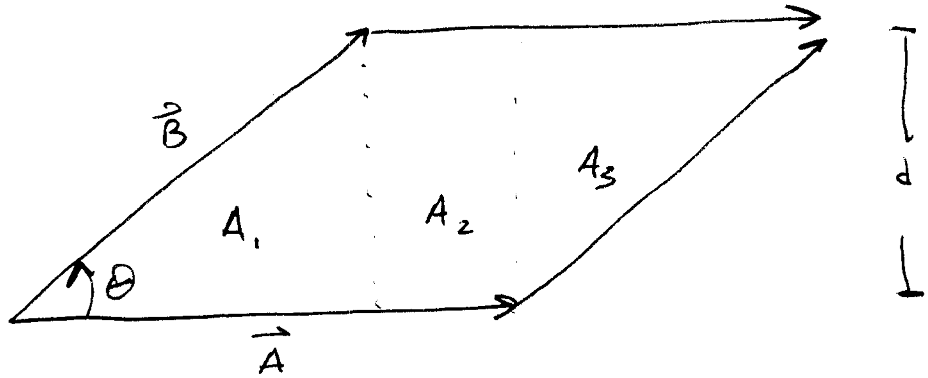
$$-(B_x C_z - B_z C_x)\,\hat{j}$$

$$+(B_x C_y - B_y C_x)\,\hat{k}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \begin{pmatrix} i & j & k \\ A_x & A_y & A_z \\ B_y C_z - B_z C_x, & -(B_x C_z - B_z C_x), & (B_x C_y - B_y C_x) \end{pmatrix}$$

$$=$$

Problem 1.10



By Combining A_1 and A_3 we form a rectangle with the same area as the parallelgram. The area of the rectangle is $d\,|\vec{A}|$ = Area

$$\frac{d}{|\vec{B}|} = \sin\theta$$

$$\text{Area} = |\vec{A}|\,|\vec{B}|\sin\theta = |\vec{A}\times\vec{B}|$$

Problem 1.11

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{pmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{pmatrix}$$

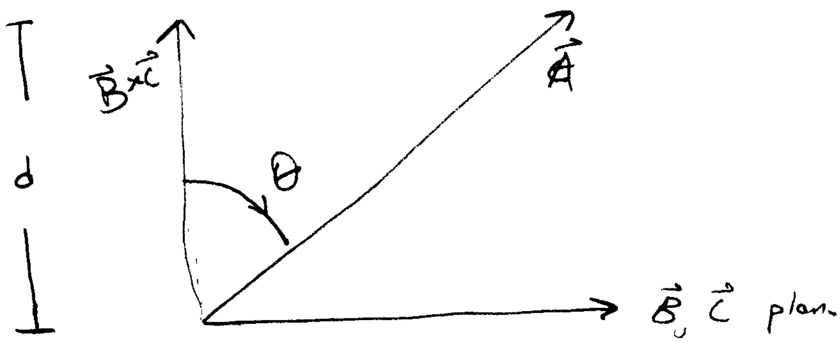
$$\cancel{= - \left(B_x \quad B_y \quad B_z \right.}$$

$$= - \begin{pmatrix} B_x & B_y & B_z \\ A_x & A_y & A_z \\ C_x & C_y & C_z \end{pmatrix}$$

$$= -\vec{B}\cdot(\vec{A}\times\vec{C})$$

Problem 1.12 Show $|\vec{A}\cdot(\vec{B}\times\vec{C})|$ is volume of parallelpiped.

$$|\vec{A}\cdot(\vec{B}\times\vec{C})| = |\vec{A}||\vec{B}\times\vec{C}|\cos\theta$$



$$\frac{d}{|\vec{A}|} = \cos\theta$$

The area of the parallel piped (imagine adding slices in the shape of the base) is $d|\vec{B}\times\vec{C}| = |\vec{A}||\vec{B}\times\vec{C}|\cos\theta$

Problem 1.13

Rotation about z - axis

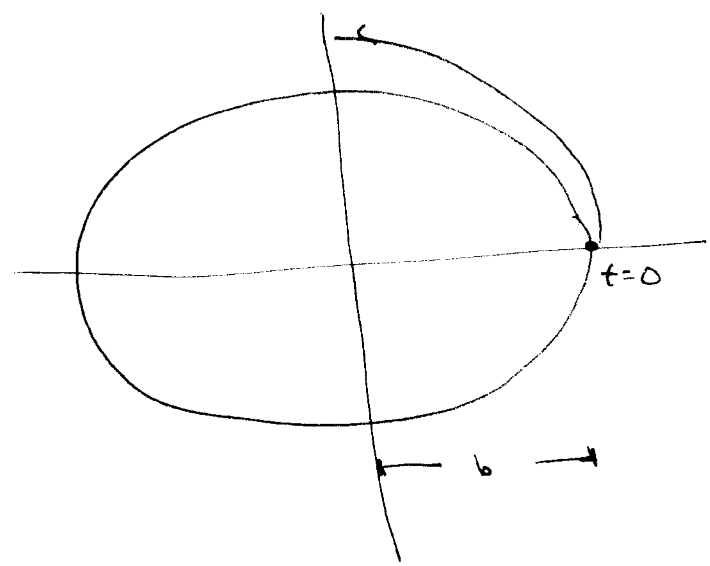
$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 1.17

Ball spun in trajectory -

$$\vec{r}(t) = \hat{i}\, b\cos\omega t + \hat{j}\, 2b\sin\omega t$$

$$\vec{v}(t) = \dot{\vec{r}} = -\hat{i}\, b\omega\sin\omega t + 2b\omega\cos\omega t\,\hat{j}$$



$$\vec{v}(0) = 2b\omega\,\hat{j}$$

$$\vec{v}\left(\frac{\pi}{2\omega}\right) = -b\omega\,\hat{i}$$

Speed

$$\vec{v} = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{b^2\omega^2\sin^2\omega t + 4b^2\omega^2\cos^2\omega t}$$

$$= \sqrt{b^2\omega^2 + 3b^2\omega^2\cos^2\omega t}$$

Problem 1.21

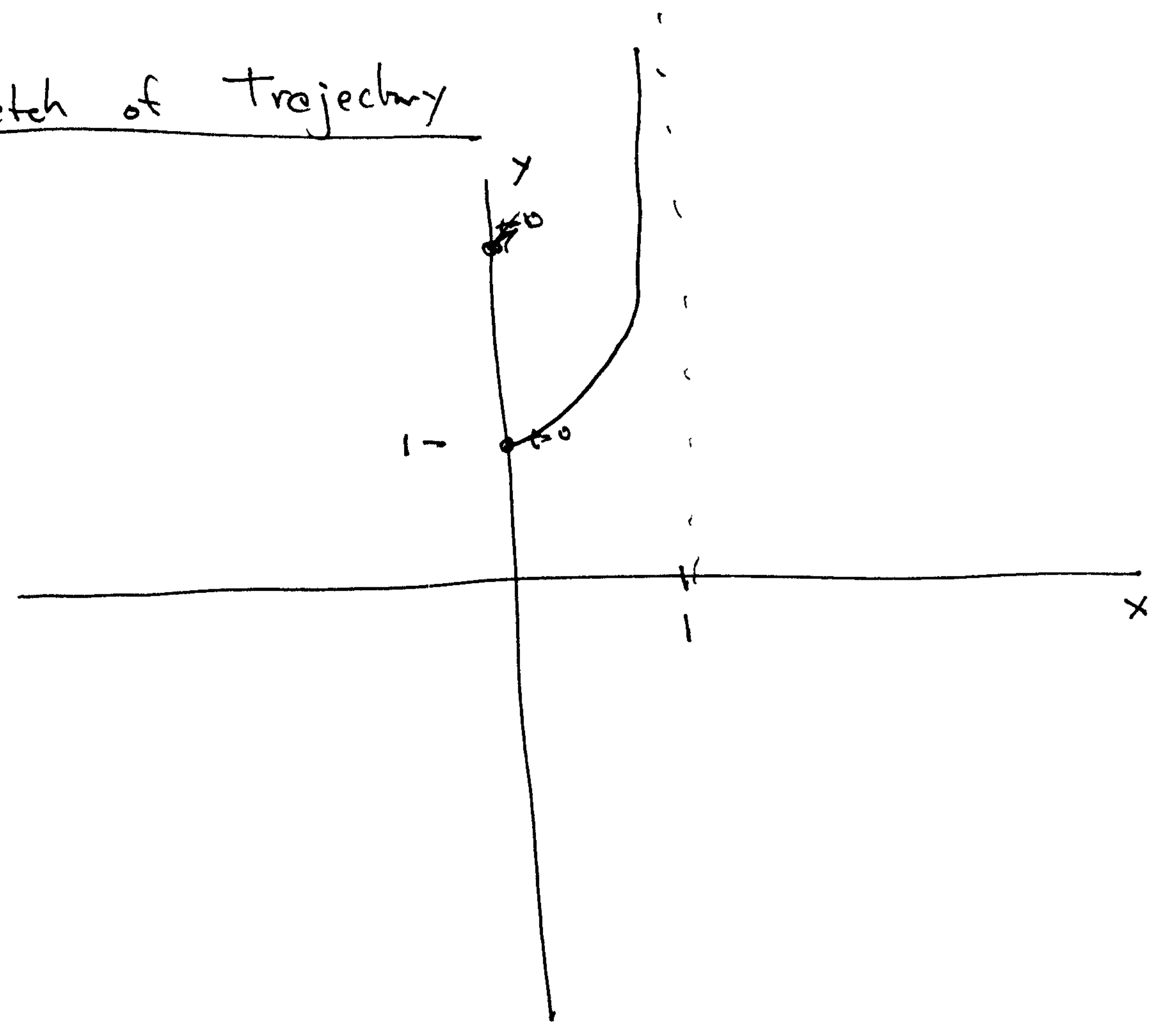
The position of a particle is given by

$$\vec{r}(t) = \hat{\imath}(1 - e^{-kt}) + \hat{\jmath} e^{kt}$$

$$\vec{v} = \dot{\vec{r}} = \hat{\imath} k e^{-kt} + \hat{\jmath} k e^{kt}$$

$$\vec{a} = -\hat{\imath} k^2 e^{-kt} + \hat{\jmath} k^2 e^{kt}$$

Sketch of Trajectory



Problem 1.24

$$\frac{d}{dt}\ \vec{r}\cdot(\vec{v}\times\vec{a})$$

$$= \overbrace{\vec{v}\cdot(\vec{v}\times\vec{a})}^{=0} + \vec{r}\cdot\frac{d}{dt}(\vec{v}\times\vec{a})$$

$$\vec{v}\times\vec{a} \perp \vec{v}$$

$$\frac{d}{dt}(\vec{v}\times\vec{a}) = \underbrace{\vec{a}\times\vec{a}}_{0} + \vec{v}\times\dot{\vec{a}}$$

$$\Rightarrow \quad \frac{d}{dt}\ \vec{r}\cdot(\vec{v}\times\vec{a}) = \vec{r}\cdot(\vec{v}\times\dot{\vec{a}})$$