Difference Between Analytic and Numerical Solutions.

$v_i := 1 - e^{-T_i}$ ← Analytic solution for linear retarding force

$u_i := \dfrac{(e^{2 \cdot T_i} - 1)}{(e^{2 \cdot T_i} + 1)}$ ← Analytic solution for quadratic retarding force

$\Delta uL_i := \dfrac{(v_i - uL_i)}{v_i}$ ← Difference, linear case

$\Delta uQ_i := \dfrac{(u_i - uQ_i)}{u_i}$ ← Difference, quadratic case

PROBLEMS

2.1 Find the velocity $\dot{x}$ and the position x as functions of the time t for a particle of mass m, which starts from rest at $x = 0$ and $t = 0$, subject to the following force functions:

(a) $F_x = F_0 + ct$

(b) $F_x = F_0 \sin ct$

(c) $F_x = F_0 e^{ct}$

where F_0 and c are positive constants.

2.2 Find the velocity $\dot{x}$ as a function of the displacement x for a particle of mass m, which starts from rest at $x = 0$, subject to the following force functions:

(a) $F_x = F_0 + cx$

(b) $F_x = F_0 e^{-cx}$

(c) $F_x = F_0 \cos cx$

where F_0 and c are positive constants.

2.3 Find the potential energy function $V(x)$ for each of the forces in Problem 2.2.

2.4 A particle of mass m is constrained to lie along a frictionless, horizontal plane subject to a force given by the expression $F(x) = -kx$. It is projected from $x = 0$ to the right along the positive x direction with initial kinetic energy $T_0 = 1/2\, kA^2$. k and A are positive constants. Find **(a)** the potential energy function $V(x)$ for this force; **(b)** the kinetic energy, and **(c)** the total energy of the particle as a function of its position. **(d)** Find the turning points of the motion. **(e)** Sketch the potential, kinetic, and total energy functions. (*Optional*: Use *Mathcad* or *Mathematica* to plot these functions. Set k and A each equal to 1.)

2.5 As in the problem above, the particle is projected to the right with initial kinetic energy T_0 but subject to a force $F(x) = -kx + kx^3/A^2$, where k and A are positive constants. Find **(a)** the potential energy function $V(x)$ for this force; **(b)** the kinetic energy, and **(c)** the total energy of the particle as a function of its position. **(d)** Find the turning points of the motion and the condition the total energy of the particle must satisfy if its motion is to exhibit turning points. **(e)** Sketch the potential, kinetic, and total energy functions. (*Optional*: Use *Mathcad* or *Mathematica* to plot these functions. Set k and A each equal to 1.)

2.6 A particle of mass m moves along a frictionless, horizontal plane with a speed given by $v(x) = \alpha/x$, where x is its distance from the origin and α is a positive constant. Find the force $F(x)$ to which the particle is subject.

2.7 A block of mass M has a string of mass m attached to it. A force $\mathbf{F}$ is applied to the string, and it pulls the block up a frictionless plane that is inclined at an angle θ to the horizontal. Find the force that the string exerts on the block.

2.8 Given that the velocity of a particle in rectilinear motion varies with the displacement x according to the equation

$$\dot{x} = bx^{-3}$$

where b is a positive constant, find the force acting on the particle as a function of x. (*Hint:* $F = m\ddot{x} = m\dot{x}\, d\dot{x}/dx$.)

2.9 A baseball (radius = .0366 m, mass = .145 kg) is dropped from rest at the top of the Empire State Building (height = 1250 ft). Calculate (**a**) the initial potential energy of the baseball, (**b**) its final kinetic energy, and (**c**) the total energy dissipated by the falling baseball by computing the line integral of the force of air resistance along the baseball's total distance of fall. Compare this last result to the difference between the baseball's initial potential energy and its final kinetic energy. (*Hint: In part (c) make approximations when evaluating the hyperbolic functions obtained in carrying out the line integral.*)

2.10 A block of wood is projected up an inclined plane with initial speed v_0. If the inclination of the plane is 30° and the coefficient of sliding friction $\mu_K = 0.1$, find the total time for the block to return to the point of projection.

2.11 A metal block of mass m slides on a horizontal surface that has been lubricated with a heavy oil so that the block suffers a viscous resistance that varies as the 3/2 power of the speed:

$$F(v) = -cv^{3/2}$$

If the initial speed of the block is v_0 at $x = 0$, show that the block cannot travel farther than $2mv_0^{1/2}/c$.

2.12 A gun is fired straight up. Assuming that the air drag on the bullet varies quadratically with speed, show that the speed varies with height according to the equations

$$v^2 = Ae^{-2kx} - \frac{g}{k} \qquad (\textit{upward motion})$$

$$v^2 = \frac{g}{k} - Be^{2kx} \qquad (\textit{downward motion})$$

in which A and B are constants of integration, g is the acceleration of gravity, and $k = c_2/m$ where c_2 is the drag constant and m is the mass of the bullet. (*Note:* x is measured positive upward, and the gravitational force is assumed to be constant.)

2.13 Use the above result to show that, when the bullet hits the ground on its return, the speed will be equal to the expression

$$\frac{v_0 v_t}{(v_0^2 + v_t^2)^{1/2}}$$

in which v_0 is the initial upward speed and

$$v_t = (mg/c_2)^{1/2} = \text{terminal speed} = (g/k)^{1/2}$$

(This result allows one to find the fraction of the initial kinetic energy lost through air friction.)

2.14 A particle of mass m is released from rest a distance b from a fixed origin of force that attracts the particle according to the inverse square law:

$$F(x) = -kx^{-2}$$

Show that the time required for the particle to reach the origin is

$$\pi\left(\frac{mb^3}{8k}\right)^{1/2}$$

2.15 Show that the terminal speed of a falling spherical object is given by

$$v_t = [(mg/c_2) + (c_1/2c_2)^2]^{1/2} - (c_1/2c_2)$$

when *both* the linear and the quadratic terms in the drag force are taken into account.

2.16 Use the above result to calculate the terminal speed of a soap bubble of mass 10^{-7} kg and diameter 10^{-2} m. Compare your value with the value obtained by using Equation 2.4.10.

2.17 Given: The force acting on a particle is the product of a function of the distance and a function of the velocity: $F(x, v) = f(x)g(v)$. Show that the differential equation of motion can be solved by integration. If the force is a product of a function of distance and a function of time, can the equation of motion be solved by simple integration? Can it be solved if the force is a product of a function of time and a function of velocity?

2.18 The force acting on a particle of mass m is given by

$$F = kvx$$

in which k is a positive constant. The particle passes through the origin with speed v_0 at time $t = 0$. Find x as a function of t.

2.19 A surface-going projectile is launched horizontally on the ocean from a stationary warship, with initial speed v_0. Assume that its propulsion system has failed and it is slowed by a retarding force given by $F(v) = -Ae^{\alpha v}$. **(a)** Find its speed as a function of time, $v(t)$. Find **(b)** the time elapsed and **(c)** the distance traveled when the projectile finally comes to rest. A and α are positive constants.

2.20 Assume that a water droplet falling though a humid atmosphere gathers up mass at a rate that is proportional to its cross-sectional area A. Assume that the droplet starts from rest and that its initial radius R_0 is so small that it suffers no resistive force. Show that **(a)** its radius and **(b)** its speed increase linearly with time.

COMPUTER PROBLEMS

C 2.1 A parachutist of mass 70 kg jumps from a plane at an altitude of 32 km above the surface of the Earth. Unfortunately, the parachute fails to open. (In the following parts, neglect horizontal motion and assume that the initial velocity is zero.)

(a) Calculate the time of fall (accurate to 1 s) until ground impact, given no air resistance and a constant value of g.

(b) Calculate the time of fall (accurate to 1 s) until ground impact, given constant g and a force of air resistance given by

$$F(v) = -c_2 v|v|$$

where c_2 is 0.5 in SI units for a falling man and is constant.

Problem 2.1

(a) $F_x = m\ddot{x} = F_0 + c\cancel{t}$

$$\cancel{F = -\frac{dU}{dx}} =$$

$$m\frac{dv}{dt} = F_0 + ct$$

$$m\,dv = (F_0 + ct)\,dt$$

$$m\int_{v_0}^{v} dv = \int_{t_0}^{t} (F_0 + ct)\,dt$$

$$m(v - v_0) = F_0(t - t_0) + \frac{c}{2}(t^2 - t_0^2)$$

$$v_0 = 0 \quad \text{at} \quad t_0 = 0$$

$$mv = F_0 t + \frac{ct^2}{2}$$

$$v(t) = \frac{1}{m}\left(F_0 t + \frac{ct^2}{2}\right) = \frac{dx}{dt}$$

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} \frac{1}{m}\left(F_0 t + \frac{ct^2}{2}\right) dt$$

$$x - x_0 = \frac{1}{m}\left(\frac{F_0}{2}(t^2 - t_0^2) + \frac{c}{6}(t^3 - t_0^3)\right)$$

$$x = \frac{1}{m}\left[\frac{F_0 t^2}{2} + \frac{ct^3}{6}\right]$$

(b)

$$F_x = m\frac{dv}{dt} = F_0 \sin ct$$

$$\int_0^v dv = \frac{F_0}{m}\int_0^t \sin ct \, dt$$

$$v(t) = -\frac{F_0}{mc}\cos ct$$

$$x(t) = \int_0^x dx = -\frac{F_0}{mc}\int_0^t \cos ct \, dt$$

$$= -\frac{F_0}{mc^2}\sin ct$$

(c) $$F_x = m\frac{dv}{dt} = F_0 e^{ct}$$

$$v(t) = \frac{F_0}{m}\int_0^t e^{ct}\, dt = \frac{F_0}{cm}e^{ct}$$

$$x(t) = \frac{F_0}{c^2 m}e^{ct}$$

2.2

(a) $F_x = -\frac{dU}{dx} = F_0 + cx$

$$F(x) = mv\frac{dv}{dx} = F_0 + cx$$

$$\int_{v_0}^{v} mv\,dv = \int_0^x (F_0 + cx)\,dx$$

$$\frac{mv^2}{2} = F_0 x + \frac{c}{2}x^2$$

$$v = \sqrt{\frac{2}{m}\left(F_0 x + \frac{c}{2}x^2\right)}$$

(b) $F(x) = mv\frac{dv}{dx} = F_0\, e^{-cx}$

$$\frac{1}{2}mv^2 = -\frac{F_0}{c}\, e^{-cx}$$

$$v = \frac{2}{m}\left[-\frac{F_0}{c}\, e^{-cx}\right]^{1/2}$$

(c) $F(x) = mv\frac{dv}{dx} = F_0 \cos cx$

$$\frac{1}{2}mv^2 = \int_0^x F_0 \cos cx = \frac{F_0}{c}\sin cx$$

$$v = \frac{2}{m}\left[\frac{F_0}{c}\sin cx\right]^{1/2}$$

2.3

(a) $F_x = -\frac{dU}{dx} = F_0 + cx$

$$U(x) = -\left[F_0 x + \frac{c}{2}x^2\right] + C \text{ by observation}$$

(b) $F_x = -\frac{dU}{dx} = F_0 e^{-cx}$

$$\int_{U(x_0)}^{U(x)} dU = -F_0 \int_{x_0}^{x} e^{-cx}\,dx$$

$$U(x) - U(x_0) = \frac{F_0}{c}\left[e^{-cx} - e^{-cx_0}\right]$$

(c) $F_x = -\frac{dU}{dx} = F_0 \cos cx$

$$\int_{U(x_0)}^{U(x)} dU = -F_0 \int_{x_0}^{x} \cos(cx)\,dx$$

$$U(x) - U(x_0) = -\frac{F_0}{c}\left[\sin cx - \sin cx_0\right]$$

2.6

$$F = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt}$$

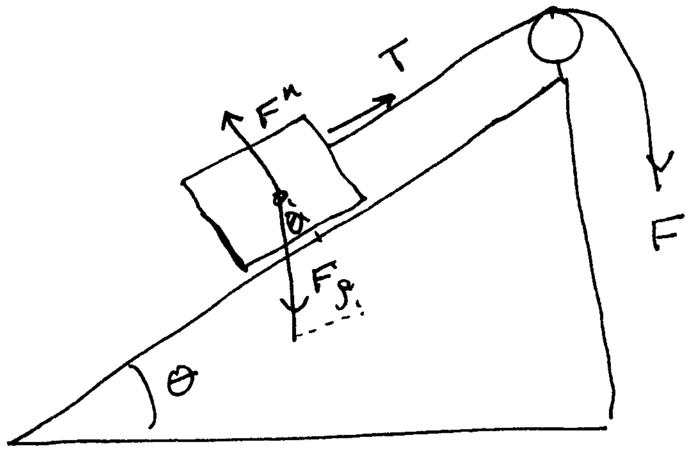
$$= mv\frac{dv}{dx}$$

Given $\quad v(x) = \dfrac{\alpha}{x}$

$$\frac{dv}{dx} = -\frac{\alpha}{x^2}$$

$$F(x) = m\left(\frac{\alpha}{x}\right)\left(\frac{-\alpha}{x^2}\right) = -\frac{m\alpha^2}{x^3}$$

2.7



Sln $F = T$

Normal $F^n - mg\cos\theta = 0$

$T - mg\sin\theta = 0$

$T = mg\sin\theta$

If the block moves with negligible velocity.

2.8

Given $$v = \frac{b}{x^3}$$

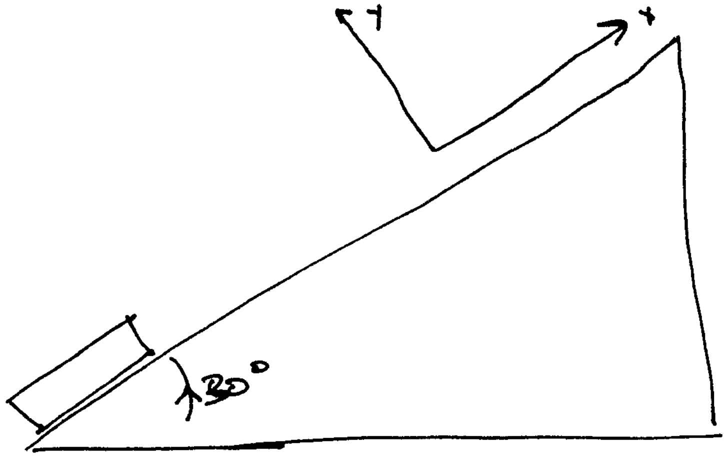
Sln $$F = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt}$$

$$= mv\frac{dv}{dx}$$

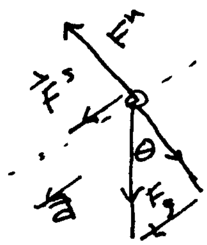
$$\frac{dv}{dx} = -\frac{3b}{x^4}$$

$$F = m\left(\frac{b}{x^5}\right)\left(-\frac{3b}{x^4}\right) = -\frac{3mb^2}{x^7}$$

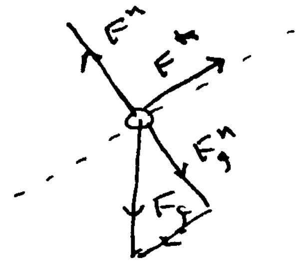
2.10



Up



Down



$$F^t_{up} = -mg\sin\theta - F^k = -mg\sin\theta - mg\mu_k\cos\theta$$

$$= -mg\left[\sin\theta + \mu_k\cos\theta\right]$$

$$F^t_{down} = -mg\left[\sin\theta - \mu_k\cos\theta\right]$$

Accelerations

$$a_{up} = -g\left[\sin\theta + \mu_k\cos\theta\right]$$

$$a_{down} = -g\left[\sin\theta - \mu_k\cos\theta\right]$$

So in both phases of the trajectory we have a constant acceleration problem.

Up Trajectory

$$v(t) = v_0 + a_{up} t \qquad = 0 \quad \text{at top}$$

$$t_{up} = -\frac{v_0}{a_{up}} = \frac{v_0}{g(\sin\theta + \mu_k \cos\theta)} = \frac{v_0}{|a_{up}|}$$

Distance travelled

$$x(t) = x_0 + v_0 t - \frac{1}{2}|a_{up}|t^2$$

$$x_0 = 0$$

$$x_f = x(t_{up}) = +\frac{v_0^2}{|a_{up}|} - \frac{1}{2}|a_{up}|\left(\frac{v_0}{a_{up}}\right)^2$$

$$= \frac{1}{2}\frac{v_0^2}{|a_{up}|} = \frac{v_0^2}{2g(\sin\theta + \mu_k(\cos\theta))}$$

Down Trajectory - The time to travel back to the starting point is the time to fall a distance

x_f ~~$x(t_{up})$~~ under an acceleration a_{down}

$$x(t) = \overset{x_f}{\cancel{x(t_{up})}} + \overset{\nearrow 0}{v_0 t} - \frac{1}{2}|a_{down}|t^2$$

$$x(t_{down}) = 0$$

$\cancel{x(t_{up}}$)

$$t_{down} = \sqrt{\frac{2x_f}{|a_{down}|}}$$

The total transit time is

$$\Delta t = t_{up} + t_{down} = \frac{v_0}{g(\sin\theta + \mu_k\cos\theta)} + \sqrt{\frac{2x_f}{|a_{down}|}}$$

$$= \frac{v_0}{g(\sin\theta + \mu_k\cos\theta)} + \left(\frac{v_0}{g}\right)\frac{1}{\sqrt{\sin^2\theta + \mu_k^2\cos^2\theta}}$$

$$\sin 30 + 0.1\cos 30 = 0.59.$$

$$\sqrt{\sin^2 30 + 0.1^2\cos^2 30} = 0.51$$

$$\Delta t = \frac{v_0}{g}\left[\frac{1}{0.59} + \frac{1}{0.51}\right]$$

$$\boxed{= 3.7\,\frac{v_0}{g}}$$

If no friction $\quad t_{up} = t_{down} \quad a = -g\sin\theta$

$$\Delta t = 2t_{up} = \frac{2v_0}{g\sin\theta} = 4\,\frac{v_0}{g}$$

2.11 Block sliding under force

$$F(v) = -cv^{3/2}$$

$$= m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx}$$

The distance travelled is —

$$-cv^{3/2} = mv\frac{dv}{dx}$$

$$\int_0^x dx = \int_{v_0}^{v} -\frac{mv}{cv^{3/2}}\,dv$$

$$x = -\frac{m}{c}\int_{v_0}^{v}\frac{dv}{\sqrt{v}} = -\frac{m}{c}\int_{v_0}^{v} v^{-1/2}\,dv$$

$$= -\frac{m}{c}\left.\frac{v^{1/2}}{1/2}\right|_{v_0}^{v}$$

$$= +\frac{2m}{c}\left(\sqrt{v_0} - \sqrt{v}\right)$$

The particle is stopped when $v=0$, so the maximum distance travelled is

$$x_{max} = \frac{2m\sqrt{v_0}}{c}$$

2.17

$$F = m\frac{dv}{dt} = mv\frac{dv}{dx}$$

So the force integral can be done for

$$F = f(x)\, g(v)$$

and

$$F = f(t)\, g(v)$$

but not for $F = f(t)\, g(x)$

If $F = f(x)\, g(v)$ the

$$F = f(x)\, g(v) = mv\frac{dv}{dx}$$

$$\int_{x_0}^{x} f(x)\, dx = \int \frac{mv\, dv}{g(v)}$$

2.18

The force is not conservative, so integrate

$$F = kxv = mv\frac{dv}{dx}$$

$$\int_{x_0}^{x} kx\,dx = \int_{v_0}^{v} dv = v - v_0$$

$$\frac{1}{2}kx^2 - \frac{1}{2}kx_0^2 = v - V_0$$

$$x_0 = 0$$

$$v = v_0 + \frac{1}{2}kx^2 = \frac{dx}{dt}$$

$$\int_0^t dt = t = \int_0^x \cancel{v_0}\,\frac{dx}{v_0 + {}^1\!/\!_2\,kx^2}$$

$$= \frac{2}{k}\int_0^x \frac{dx}{\frac{2v_0}{k} + x^2}$$

$$= \frac{2}{k}\left[\sqrt{\frac{k}{2v_0}}\tan^{-1}\left(\frac{x}{\sqrt{\frac{2v_0}{k}}}\right)\right]_0^x$$

$$= \sqrt{\frac{2}{kv_0}}\tan^{-1}\left(\frac{x}{\sqrt{\frac{2v_0}{k}}}\right)$$

Let $\alpha = \sqrt{\frac{2v_0}{k}}$ the $t = \frac{2}{k\alpha}\tan^{-1}\left(\frac{x}{\alpha}\right)$

$$\frac{k\alpha t}{2} = \tan^{-1}\left(\frac{x}{\alpha}\right)$$

$$x(t) = \alpha \tan\left[\frac{k\alpha t}{2}\right]$$

2.19

$$F(v) = -Ae^{\alpha v} = mv\frac{dv}{dx} = m\frac{dv}{dt}$$

Speed as Function of Time

$$dt = -\frac{m}{A}\frac{dv}{e^{\alpha v}} = -\frac{m}{A}e^{-\alpha v}\,dv$$

$$t - t_0 = -\frac{m}{\cancel{A}}\int_{v_0}^{v} e^{-\alpha v}\,dv \qquad u = -\alpha v$$

$$= \frac{m}{\alpha A}\int_{-\alpha v_0}^{-\alpha v} e^{u}\,du$$

$$t - t_0 = \frac{m}{\alpha A}\left[e^{-\alpha v} - e^{-\alpha v_0}\right]$$

$$\frac{\alpha A}{m}(t - t_0) + e^{-\alpha v_0} = e^{-\alpha v}$$

$$v = -\frac{1}{\alpha}\ln\left[\frac{\alpha A}{m}(t - t_0) + e^{-\alpha v_0}\right]$$

The partical comes to a halt ~~after~~ when $v = 0$

$$t_f = \frac{m}{\alpha A}\left[1 - e^{-\alpha v_0}\right] \qquad \text{if } t_0 = 0$$

Solve for the trajectory as a function of time

$$mv\frac{dv}{dx} = -Ae^{av}$$

$$\int_0^x dx = x = -\frac{m}{A}\int_{v_0}^{v} \frac{v\,dv}{e^{av}} =$$

$$u = -av \qquad du = -a\,dv$$

$$x = \frac{-m}{Aa^2}\int_{u_0}^{u} u e^{u} du$$

<u>Integrate by parts</u>

$$u = x \qquad du = dx$$

$$e^u du = dy \qquad e^u = y$$

$$d(xy) = x\,dy + y\,dx$$

$$\int x\,dy = xy - \int y\,dx$$

$$\int u e^u du = ue^u - \int e^u du$$

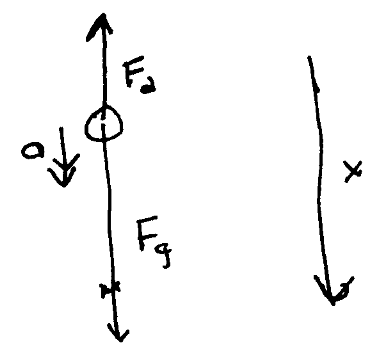
$$= \cancel{ue^u}\ (u-1)e^u$$

$$x = \frac{-m}{Aa^2}\left[u-1\right]e^u \Big|_{-av_0}^{-av} = \frac{-m}{Aa^2} + \text{mess}$$

Section 4.5 - Example

A sphere is dropped from a height h_0. Quadratic drag dominates. Find everything.

(1) <u>Free Body Diagram</u>



(2) Terminal Velocity

$$\dot{v} = 0 \implies F = 0$$

$$F = mg - c_2 v_t^2 = 0$$

$$v_t = \sqrt{\frac{mg}{c_2}}$$

(3) $v(x)$ -

$$F = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx}$$

$$= mg - c_2 v^2$$

$$\int_0^x dx = x = \int_0^v \frac{mv\,dv}{mg - c_2 v^2}$$

Let $u = mg - c_2 v^2$

$$du = -2c_2 v\,dv$$

$$v\,dv = \frac{du}{-2c_2}$$

$$x = -\frac{m}{2c_2}\int_{u_0}^{u} \frac{du}{u}$$

$$= -\frac{m}{2c_2} \ln u/u_0$$

$$u_0 \exp\left[-\frac{2c_2}{m}x\right] = u$$

$$u_0 = mg$$

$$mg \exp\left[-\frac{2c_2}{m}x\right] = mg - c_2 v^2$$

$$v^2 = \frac{mg}{c_2}\left[1 - \exp\left(-\frac{2c_2}{m}x\right)\right]$$

$$= v_t^2\left[1 - \exp\left(-\frac{2c_2}{m}x\right)\right]$$

Characteristic Distance to achieve terminal velocity

$$\lambda_t = \frac{m}{2c_2}$$

(4) velocity as function of time

$$F = m\frac{dv}{dt} = mg - c_2 v^2$$

(5) ~~Power Dissipated by friction~~ Work done by Drag

$$\cancel{P = F_{nm} v}$$

$$\Delta E_{sys} = \int F_{nm}\, dx = \int c_2 v^2\, dv \quad *$$

$$= \cancel{U_f}\; U(h_0) - T(v_t)$$

$$\boxed{\text{Ignore hint on 2.9, use above.}}$$