$$F_{ext}(t) = F_0\left[0.1 + \frac{2}{\pi}\sin(0.1\pi)\cos(\omega t) + \frac{2}{2\pi}\sin(0.2\pi)\cos(2\omega t) + \frac{2}{3\pi}\sin(0.3\pi)\cos(3\omega t) + \ldots\right]$$

$$= F_0[0.1 + 0.197\cos(\omega t) + 0.187\cos(2\omega t) + 0.172\cos(3\omega t) + \ldots]$$

The resonance denominators in Equation 3.9.14 are given by

$$D_n = \left[\left(\omega_0^2 - n^2\frac{\omega_0^2}{4}\right)^2 + 4(0.1)^2\omega_0^2 n^2\frac{\omega_0^2}{4}\right]^{1/2} = \left[\left(1 - \frac{n^2}{4}\right)^2 + 0.01n^2\right]^{1/2}\omega_0^2$$

Thus,

$$D_0 = \omega_0^2 \qquad D_1 = 0.757\omega_0^2 \qquad D_2 = 0.2\omega_0^2 \qquad D_3 = 1.285\omega_0^2$$

The phase angles (Equation 3.9.15) are

$$\phi_n = \tan^{-1}\left(\frac{0.2n\omega_0^2/2}{\omega_0^2 - n^2\omega_0^2/4}\right) = \tan^{-1}\left(\frac{0.4n}{4 - n^2}\right)$$

which gives

$$\phi_0 = 0 \qquad \phi_1 = \tan^{-1}(0.133) = 0.132$$
$$\phi_2 = \tan^{-1}\infty = \pi/2 \qquad \phi_3 = \tan^{-1}(-0.24) = -0.236$$

The steady-state motion of the system is therefore given by the following series (Equation 3.9.13):

$$x(t) = \frac{F_0}{m\omega_0^2}[0.1 + 0.26\cos(\omega t - 0.132) + 0.935\sin(2\omega t) + 0.134\cos(3\omega t + 0.236) + \ldots$$

The dominant term is the one involving the second harmonic $2\omega = \omega_0$, because ω_0 is close to the resonant frequency. Note also the phase of this term: $\cos(2\omega t - \pi/2) = \sin(2\omega t)$.

PROBLEMS

3.1 A guitar string vibrates harmonically with a frequency of 512 Hz (one octave above middle C on the musical scale). If the amplitude of oscillation of the centerpoint of the string is 0.002 m (2 mm), what are the maximum speed and the maximum acceler- at that point?

3.2 A piston executes simple harmonic motion with an amplitude of 0.1 m. If it passes through the center of its motion with a speed of 0.5 m/s, what is the period of oscillation?

3.3 A particle undergoes simple harmonic motion with a frequency of 10 Hz. Find the dis- placement x at any time t for the following initial condition:

$$t = 0 \qquad x = 0.25\ \text{m} \qquad \dot{x} = 0.1\ \text{m/s}$$

3.4 Verify the relations among the four quantities C, D, ϕ_0, and A given by Equation 3.2.19.

3.5 A particle undergoing simple harmonic motion has a velocity $\dot{x}_1$ when the displacement is x_1 and a velocity $\dot{x}_2$ when the displacement is x_2. Find the angular frequency and the amplitude of the motion in terms of the given quantities.

3.6 On the surface of the moon, the acceleration of gravity is about one-sixth that on the Earth. What is the half-period of a simple pendulum of length 1 m on the moon?

3.7 Two springs having stiffness k_1 and k_2, respectively, are used in a vertical position to support a single object of mass m. Show that the angular frequency of oscillation is $[(k_1 + k_2)/m]^{1/2}$ if the springs are tied in parallel, and $[k_1 k_2/(k_1 + k_2)m]^{1/2}$ if the springs are tied in series.

3.8 A spring of stiffness k supports a box of mass M in which is placed a block of mass m. If the system is pulled downward a distance d from the equilibrium position and then released, find the force of reaction between the block and the bottom of the box as a function of time. For what value of d will the block just begin to leave the bottom of the box at the top of the vertical oscillations? Neglect any air resistance.

3.9 Show that the ratio of two successive maxima in the displacement of a damped harmonic oscillator is constant. (*Note:* The maxima do not occur at the points of contact of the displacement curve with the curve $Ae^{-\gamma t}$.)

3.10 A damped harmonic oscillator with $m = 10$ kg, $k = 250$ N/m, and $c = 60$ kg/s is subject to a driving force given by $F_0 \cos \omega t$, where $F_0 = 48$ N.

(a) What value of ω results in steady-state oscillations with maximum amplitude? Under this condition:

(b) What is the maximum amplitude?

(c) What is the phase shift?

3.11 A mass m moves along the x-axis subject to an attractive force given by $17\beta^2 mx/2$ and a retarding force given by $3\beta m\dot{x}$, where x is its distance from the origin and β is a constant. A driving force given by $mA \cos \omega t$, where A is a constant, is applied to the particle along the x-axis.

(a) What value of ω results in steady-state oscillations about the origin with maximum amplitude?

(b) What is the maximum amplitude?

3.12 The frequency f_d of a damped harmonic oscillator is 100 Hz, and the ratio of the amplitude of two successive maxima is one half.

(a) What is the undamped frequency f_0 of this oscillator?

(b) What is the resonant frequency f_r?

3.13 Given: The amplitude of a damped harmonic oscillator drops to $1/e$ of its initial value after n complete cycles. Show that the ratio of period of oscillation to the period of the same oscillator with no damping is given by

$$\frac{T_d}{T_0} = \left(1 + \frac{1}{4\pi^2 n^2}\right)^{1/2} \approx 1 + \frac{1}{8\pi^2 n^2}$$

where the approximation in the last expression is valid if n is large. (See the approximation formulas in Appendix D.)

3.14 Work all parts of Example 3.6.1 for the case in which the exponential damping factor γ is one-half the critical value and the driving frequency is equal to $2\omega_0$.

3.15 For a lightly damped harmonic oscillator $\gamma \ll \omega_0$, show that the driving frequency for which the steady-state amplitude is one-half the steady-state amplitude at the resonant frequency is given by $\omega \approx \omega_0 \pm \gamma\sqrt{3}$.

3.16 If a series LCR circuit is connected across the terminals of an electric generator that produces a voltage $V = V_0 e^{i\omega t}$, the flow of electrical charge q through the circuit is given by the following second-order differential equation:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V_0 e^{i\omega t}$$

(a) Verify the correspondence shown in Table 3.6.1 between the parameters of a driven mechanical oscillator and the above driven electrical oscillator.

(b) Calculate the Q of the electrical circuit in terms of the coefficients of the above differential equation.

(c) Show that, in the case of small damping, Q can be written as $Q = R_0/R$, where $R_0 = \sqrt{L/C}$ is the *characteristic impedance* of the circuit.

3.17 A damped harmonic oscillator is driven by an external force of the form

$$F_{ext} = F_0 \sin \omega t$$

Show that the steady-state solution is given by

$$x(t) = A(\omega) \sin(\omega t - \phi)$$

where $A(\omega)$ and ϕ are identical to the expressions given by Equations 3.6.7e and 3.6.8.

3.18 Solve the differential equation of motion of the damped harmonic oscillator driven by a damped harmonic force:

$$F_{ext}(t) = F_0 e^{-\alpha t} \cos \omega t$$

(Hint: $e^{-\alpha t}\cos \omega t = Re(e^{-\alpha t + i\omega t}) = Re(e^{\beta t})$, *where* $\beta = -\alpha + i\omega$. *Assume a solution of the form* $Ae^{\beta t - i\phi}$.*)*

3.19 A simple pendulum of length l oscillates with an amplitude of 45°.

(a) What is the period?

(b) If this pendulum is used as a laboratory experiment to determine the value of g, find the error included in the use of the elementary formula $T_0 = 2\pi(l/g)^{1/2}$.

(c) Find the approximate amount of third-harmonic content in the oscillation of the pendulum.

3.20 Verify Equations 3.9.9 and 3.9.10 in the text.

3.21 Show that the Fourier series for a periodic square wave is

$$f(t) = \frac{4}{\pi}\left[\sin(\omega t) + \frac{1}{3}\sin(3\omega t) + \frac{1}{5}\sin(5\omega t) + \cdots\right]$$

where

$$f(t) = +1 \qquad \text{for } 0 < \omega t < \pi,\ 2\pi < \omega t < 3\pi, \text{ and so on}$$
$$f(t) = -1 \qquad \text{for } \pi < \omega t < 2\pi,\ 3\pi < \omega t < 4\pi, \text{ and so on}$$

3.22 Use the above result to find the steady-state motion of a damped harmonic oscillator that is driven by a periodic square-wave force of amplitude F_0. In particular, find the relative amplitudes of the first three terms, A_1, A_3, and A_5 of the response function $x(t)$ in the case that the third harmonic 3ω of the driving frequency coincides with the frequency ω_0 of the undamped oscillator. Let the quality factor $Q = 100$.

3.23 **(a)** Derive the first-order differential equation, dy/dx, describing the phase-space trajectory of the simple harmonic oscillator.

(b) Solve the equation, proving that the trajectory is an ellipse.

3.24 A simple pendulum whose length $l = 9.8$ m satisfies the equation

$$\ddot{\theta} + \sin\theta = 0$$

(a) If Θ_0 is the amplitude of oscillation, show that its period T is given by

$$T = 4\int_0^{\pi/2} \frac{d\phi}{(1 - \alpha\sin^2\phi)^{1/2}} \qquad \text{where } \alpha = \sin^2\frac{1}{2}\Theta_0$$

(b) Expand the integrand in powers of α, integrate term by term, and find the period T as a power series in α. Keep terms up to and including $O(\alpha^2)$.

(c) Expand α in a power series of Θ_0, insert the result into the power series found in (b), and find the period T as a power series in Θ_0. Keep terms up to and including $O(\Theta_0^2)$.

COMPUTER PROBLEMS

C 3.1 The exact equation of motion for a simple pendulum of length L (see Example 3.2.2) is given by

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0$$

where $\omega_0^2 = g/L$. Find $\theta(t)$ by numerically integrating this equation of motion. Let $L = 1.00$ m. Let the initial conditions be $\theta_0 = \pi/2$ rad and $\dot{\theta}_0 = 0$ rad/s.

(a) Plot $\theta(t)$ from $t = 0$ to 4 s. Also, plot the solution obtained by using the small-angle approximation ($\sin\theta \approx \theta$) on the same graph.

(b) Repeat (a) for $\theta_0 = 3.10$ rad.

(c) Plot the period of the pendulum as a function of the amplitude θ_0 from 0 to 3.10 rad. At what amplitude does the period deviate by more than 2% from $\sqrt{g/L}$?

C 3.2 Assume that the damping force for the damped harmonic oscillator is proportional to the square of its velocity; that is, it is given by $-c_2\dot{x}|\dot{x}|$. The equation of motion for such an oscillator is thus

$$\ddot{x} + 2\gamma\dot{x}|\dot{x}| + \omega_0^2 x = 0$$

where $\gamma = c_2/2m$ and $\omega_0^2 = k/m$. Find $x(t)$ by numerically integrating the above equation of motion. Let $\gamma = 0.20\ \text{m}^{-1}\,\text{s}^{-1}$ and $\omega_0 = 2.00$ rad/s. Let the initial conditions be $x(0) = 1.00$ m and $\dot{x}(0) = 0$ m/s.

(a) Plot $x(t)$ from $t = 0$ to 20 s. Also, on the same graph, plot the solution for the damped harmonic oscillator where the damping force is linearly proportional to the velocity; that is, it is given by $-c_1\dot{x}$. Again, let $\gamma = c_1/2m = 0.20\ \text{s}^{-1}$ and $\omega_0 = 2.00$ rad/s.

(b) For the case of linear damping, plot the log of the absolute value of the successive extrema versus their time of occurrence. Find the slope of this plot, and use it to estimate γ. (This method works well for the case of weak damping.)

3.1

$f = 512\ s^{-1}$ $\qquad$ $A = 0.002\,m$

$$x(t) = A \sin \omega t$$

$$\omega = 2\pi f$$

$$\dot{x}(t) = A\omega \cos \omega t$$

$$\dot{x}_{max} = A\omega = (0.002m)(2\pi)(512\,s^{-1}) = 6.43\ m/s$$

$$\ddot{x} = -A\omega^2 \sin \omega t$$

$$\ddot{x}_{max} = A\omega^2 = (0.002m)(2\pi\ 512\,s^{-1})^2$$

$$= 2 \times 10^4\ m/s^2$$

3.3

$f = 10\ s^{-1}$

$t = 0 \qquad x = 0.25m \qquad \dot{x} = 0.1\ m/s$

$\omega = 2\pi f = 20\pi\ s^{-1}$

$x(t) = A \sin(\omega t + \phi)$

$x(0) = A \sin\phi = x_0$

$\dot{x} = A\omega \cos(\omega t + \phi)$

$\dot{x}(0) = A\omega \cos\phi = v_0$

$$\sin\phi = \frac{x_0}{A}$$

$$\cos\phi = \frac{v_0}{A\omega}$$

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{x_0 \omega}{v_0} = \frac{(0.25m)(2\pi)(10s^{-1})}{0.1\ m/s}$$

$=$

$\phi = \cancel{15}\ 89.6^\circ$

$$A = \frac{x_0}{\sin\phi} = 0.1000002$$

My choice of representation was wrong.

$$x(t) = C\cos\omega_0 t + D\sin\omega_0 t$$

$$\dot{x}(t) = -C\omega_0 \sin\omega_0 t + D(\cos\omega_0 t)\,\omega_0$$

$$x(0) = C = 0.25\,m$$

$$\dot{x}(0) = D\omega_0 = 0.1\,m/s$$

$$D = \frac{0.1\,m/s}{\omega_0} = \frac{0.1\,m/s}{2\pi\, 10\, s^{-1}} = 0.00159\,m$$

$$x(t) = 0.25\cos(20\pi s^{-1})t + 0.00159\,m \sin(20\pi s^{-1})t$$

3.4

$$x(t) = A\sin(\omega_0 t + \phi_0)$$

$$= C\cos\omega_0 t + D\sin\omega_0 t$$

Trig Ident

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$A\sin(\omega_0 t + \phi_0) = A(\sin\omega_0 t\cos\phi + \cos\omega_0 t\sin\phi$$

$$\cancel{C = A\cos\phi} \quad \cancel{D}$$

$$C = A\sin\phi \qquad D = A\cos\phi$$

$$\frac{C}{D} = \tan\phi$$

$$C^2 + D^2 = A^2(\sin^2\phi + \cos^2\phi) = A^2$$

(3.5)

When x_1 is displacement $\dot{x}_1$ is velocity

x_2 is displacement $\dot{x}_2$ is velocity

$$x(t) = C\cos\omega_0 t + D\sin\omega_0 t$$

$$\dot{x}(t) = -C\omega_0\sin\omega_0 t + D\omega_0\cos\omega_0 t$$

At t_1, $\dot{x}_1 = -C\omega_0\sin\omega_0 t_1 + D\omega_0\cos\omega_0 t_1$

$$x_1 = C\cos\omega_0 t + D\sin\omega_0 t$$

Try other representation,

$$x(t) = A\sin(\omega t+\phi)$$

$$\dot{x}(t) = A\omega\cos(\omega t+\phi)$$

$$x(t)^2 + \frac{\dot{x}(t)^2}{\omega^2} = A^2\sin^2(\omega t+\phi) + A^2\cos^2(\omega t+\phi)$$

$$= A^2$$

$$x_1^2 + \frac{\dot{x}_1^2}{\omega^2} = A^2 \quad \Big| \quad x_2^2 + \frac{\dot{x}_2}{\omega^2} = A^2$$

$$\cancel{x_2^2 + \frac{\dot{x}_2^2}{\omega^2} =}$$

$$A = \sqrt{x_1^2 + \frac{\dot{x}_1^2}{\omega^2}}$$

$$x_1^2 + \frac{\dot{x}_1^2}{\omega^2} = x_2^2 + \frac{\dot{x}_2}{\omega^2}$$

$$x_1^2 - x_2^2 = \frac{1}{\omega^2}(\dot{x}_2 - \dot{x}_1)$$

$$\omega^2 = \frac{\dot{x}_2 - \dot{x}_1}{x_1^2 - x_2^2}$$

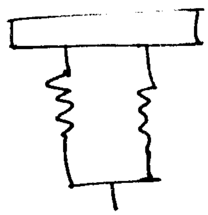
Substitute ω^2 into either eqn.

3.7

Replace Spring with equivalent spring that exerts same force.

Angular Frequency $\omega_0 = \sqrt{\frac{k}{m}}$

Parallel

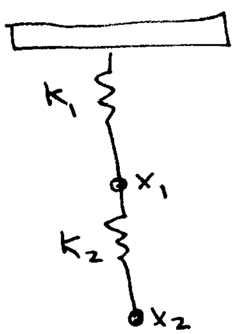


Force exerted $F = -k_1 x - k_2 x$

Equivalent spring constant $k_1 + k_2$

$$\omega_0 = \sqrt{\frac{k_1 + k_2}{m}}$$

Series



$F_1 = F_2$ for equilibrium

$$k_1 x_1 = k_2 (x_2 - x_1) = k_2 x_2 - k_2 x_1$$

$$-F = +k_2 (x_2 - x_1) = k x_2$$

$$= k_2 x_2 - k_2 x_1$$

3.7(b)

$$(k_1 + k_2)\, x_1 = k_2 x_2$$

$$x_1 = \frac{k_2 x_2}{k_1 + k_2}$$

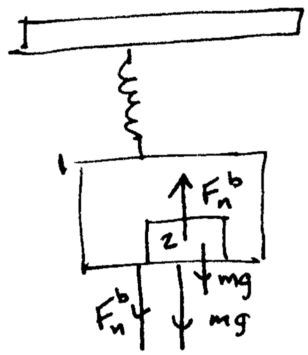
$$k x_2 = k_2 x_2 - \frac{k_2^2 x_2}{k_1 + k_2}$$

$$= \frac{k_1 k_2 x_2 + k_2^2 x_2 - k_2 x_2^2}{k_1 + k_2} = \frac{k_1 k_2}{k_1 + k_2} x_2$$

$$k = \frac{k_1 k_2}{k_1 + k_2}$$

$$\omega_0 = \sqrt{\frac{1}{m}\left(\frac{k_1 k_2}{k_1 + k_2}\right)}$$

3.8



System

The two blocks accelerate together until the ~~first~~ second block takes off.

$$a_1 = a_2$$

$$F_1 = -Mg - F_n^b - kx = M a_1$$

$$F_2 = -mg + F_n^b = m a_2$$

Add: $-(M+m)g - kx = (M+m)a$

A fixed weight only displaces equilibrium position, so frequency is the same

$$\omega_0 = \sqrt{\frac{k}{M+m}}$$

Initial condition $x(0) = d$ $\quad \dot{x}(0) = 0$

$$x(t) = d \cos(\omega_0 t)$$

$$\dot{x}(t) = -d\omega_0 \sin \omega t$$

$$\ddot{x}(t) = -d\omega_0^2 \cos \omega t$$

$$F_z = -mg + F_n^b = m\ddot{x}$$

So the block feels no reaction force when

$$\ddot{x} = g = d\omega_0^2$$

$$\cancel{\omega_0}\ d = \frac{g}{\omega_0^2} = \frac{(m+M)\,g}{k}$$

3.9

The equation of motion for an underdamped oscillator is —

$$x(t) = A e^{-\gamma t} \cos(\omega_d t + \phi_0)$$

The maxima occure when $\dot{x} = 0$

$$\dot{x}(t) = -\gamma A e^{-\gamma t} \cos(\omega_d + \phi_0) - A e^{-\gamma t} \sin(\omega_d t + \phi_0)\omega_d$$

$$= 0$$

$$\gamma \cos(\omega_d t + \phi_0) + \omega_0 \sin(\omega_d t + \phi_0) = 0$$

$$-\frac{\gamma}{\omega_d} = \tan(\omega_d t_{max} + \phi_0)$$

Let t_0 be the first extrema

$$t_0 = \frac{1}{\omega_d}\left[\tan^{-1}\left[\frac{-\gamma}{\omega_d}\right] - \phi_0\right]$$

Successive extrema

$$t_n = t_1 + \frac{2\pi n}{\omega_d}$$

$$t_{n+1} = t_n + \frac{2\pi}{\omega_d}$$

$$\frac{x(t_{n+1})}{x(t_n)} = \underbrace{\frac{A e^{-\gamma t_{n+1}}}{A e^{-\gamma t_n}}}\ \frac{\cos(\omega_d t_{n+1} + \phi_0)}{\cos(\omega_d t_n + \phi_0)}$$

$$e^{-\gamma(t_{n+1} - t_n)} = e^{-\gamma 2\pi/\omega_d}$$

$$\cos(\omega_d t_{n+1} + \phi_0) = \cos(\omega_d t_n + 2\pi + \phi_0) = \cos(\omega_d t_n + \phi_0)$$

So the ratio of the cosines vanish,

$$\frac{x(t_{n+1})}{x(t_n)} = e^{-\frac{2\pi\gamma}{\omega_d}}$$

3.11

$$m\ddot{x} = F = -\frac{17\beta^2 m x}{2} - 3\beta m\dot{x} + mA\cos\omega t$$

$$\ddot{x} + \frac{17\beta^2}{2}x + 3\beta\dot{x} = A\cos\omega t$$

$$\gamma = \tfrac{3}{2}\beta$$

$$\omega_0^2 = \frac{17\beta^2}{2}$$

<u>Resonant Frequency</u>

$$\omega_r^2 = \omega_0^2 - 2\gamma^2$$

$$= \frac{17\beta^2}{2} - \cancel{8}\left(\frac{9}{4}\right)\beta^2$$

$$= \frac{8}{2}\beta^2 = 4\beta^2$$

$$\omega_r = 2\beta$$

<u>Amplitude at Resonance</u>

$$A_{max} = \frac{F_0/m}{2\gamma\sqrt{\omega_0^2-\gamma^2}}$$

$$= \cancel{\frac{F_0/m}{2\gamma\sqrt{\;}}}$$

$$A_{max} = \frac{A}{2\left(\frac{3}{2}\beta\right)\left(\frac{17\beta^2}{2} - \frac{9}{4}\beta^2\right)^{1/2}}$$

$$= \frac{A}{3\beta} \frac{1}{\left(\frac{25\beta^2}{4}\right)^{1/2}}$$

$$= \frac{A}{3\beta} \frac{1}{\frac{5}{2}\beta} = \frac{2A}{15\beta^2}$$

3.13

$$\frac{A_n}{A} = e^{-1} = e^{-\gamma n T_d}$$

$$1 = \gamma n T_d$$

$$\frac{1}{\gamma n} = T_d$$

$$\gamma n = \frac{1}{T_d} = f_d$$

$$2\pi\gamma n = \omega_d$$

$$\frac{T_d}{T_0} = \frac{\omega_0}{\omega_d}$$

$$\omega_d^2 = \omega_0^2 - \gamma^2$$

$$\omega_0 = \sqrt{\omega_d^2 + \gamma^2}$$

$$\frac{T_d}{T_0} = \frac{\omega_0}{\omega_d} = \frac{\sqrt{\omega_d^2 + \gamma^2}}{\omega_d} = \left(1 + \frac{\gamma^2}{\omega_d^2}\right)^{1/2}$$

$$\frac{\gamma}{\omega_d} = \frac{1}{2\pi n}$$

$$\frac{T_d}{T_0} = \left(1 + \frac{1}{4\pi^2 n^2}\right)^{1/2}$$

3.15

In the limit of weak damping

$$A(\omega) = \frac{A_{max}\,\gamma}{\sqrt{(\omega_0^2-\omega)^2+\gamma^2}} = \frac{1}{2} A_{max}$$

$$2\gamma = \sqrt{(\omega_0-\omega)^2+\gamma^2}$$

$$4\gamma^2 = (\omega_0-\omega)^2+\gamma^2$$

$$3\gamma^2 = (\omega_0-\omega)^2$$

$$\omega = \omega_0 \pm \sqrt{3}\,\gamma^2$$

3.17

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F}{m}\sin(\omega t)$$

$$= -i\frac{F}{m}e^{i\omega t}$$

Try $x(t) = A e^{i(\omega t + \phi)}$

$$-\omega^2 A e^{i(\omega t+\phi)} + 2\gamma i\omega A e^{i(\omega t+\phi)} + \omega_0^2 A e^{i(\omega t+\phi)}$$

$$= -i\frac{F}{m}e^{i\omega t}$$

$$-\omega^2 A e^{i\phi} + 2\gamma i\omega A e^{i\phi} + \omega_0^2 A e^{i\phi}$$

$$= -\frac{iF}{m}$$

$$-\omega^2 A + 2\gamma i\omega A + \omega_0^2 A = -\frac{iF}{m}e^{-i\phi}$$

$$= -\frac{iF}{m}(\cos\phi - i\sin\phi)$$

$$= \frac{F}{m}(\sin\phi - i\cos\phi)$$

Real Part $\quad -\omega^2 A + \omega_0^2 A = \frac{F}{m}\sin\phi$

Im Part $\quad 2\gamma\omega A = -\frac{F}{m}\cos\phi$

$$(\omega_0^2 - \omega^2)^2 A^2 + 4\gamma^2\omega^2 A^2 = \left(\frac{F}{m}\right)^2(\sin^2\phi + \cos^2\phi)$$

$$= (F/m)^2$$

3.17(b)

$$A(\omega) = \frac{F/m}{\sqrt{(\omega_0 - \omega)^2 + 4\gamma^2\omega^2}}$$

$$\tan\phi = -\frac{(\omega_0^2 - \omega^2)}{2\gamma\omega}$$

This must carry appropriate phase angle

Now, try using imaginary part of solution

$$F_{ext} = \mathrm{Im}\left[F_0 e^{i\omega t} \right]$$

$$x(t) = \mathrm{Im}\left[A e^{i(\omega t - \phi)} \right]$$

Do the whole thing again,

$$-\cancel{6}\omega^2 A e^{-i\phi} + 2\gamma i\omega A e^{-i\phi} + \omega_0^2 A e^{-i\phi} = \frac{F_0}{m}$$

$$-\omega^2 A + 2\gamma i\omega A + \omega_0^2 A = \frac{F_0}{m} e^{i\phi}$$

$$= \frac{F_0}{m}(\cos\phi + i\sin\phi)$$

Re $\quad (\omega_0^2 - \omega^2)A = \frac{F_0}{m}\cos\phi$

Im $\quad 2\gamma\omega A = \frac{F}{m}\sin\phi$

3.17(c)

$$(\omega_0^2-\omega^2)^2A^2 + 4\gamma^2\omega^2A^2 = \left(\frac{F}{m}\right)^2(\cos^2\phi+\sin^2\phi)$$
$$= \left(\frac{F}{m}\right)^2$$

$$A(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}}$$

$$\tan\phi = \frac{2\gamma\omega}{\omega_0^2-\omega^2}$$

3.18

$$F_{ext}(t) = F_0 e^{-\alpha t} \cos\omega t$$

$$F_{ext}(t) = Re\left[e^{i\omega t - \alpha t} \right]$$

Let $B = \cancel{\omega} \ -\alpha + i\omega$

$$F_{ext}(t) = e^{Bt}$$

Try solution $\cancel{x(t) = A e^{Bt - i\phi}}$

$$x(t) = Re\left[A e^{+Bt - i\phi} \right]$$

Eqn of Motion

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{Bt}$$

$$B^2 A + 2\gamma B A + \omega_0^2 A = \frac{F_0}{m} e^{i\phi}$$

$$B^2 = (-\alpha + i\omega)^2 = \alpha^2 - 2i\omega\alpha - \omega^2$$

$$(\alpha^2 - 2i\omega\alpha - \omega^2)A + 2\gamma(i\omega - \alpha)A + \omega_0^2 A$$

$$= \frac{F_0}{m}\left(\cos\phi + i\sin\phi\right)$$

Re $\quad A(\alpha^2 - \omega^2) \cancel{+} 2\gamma\alpha A + \omega_0^2 A = \frac{F_0}{m}\cos\phi$

Im $\quad -2\omega\alpha A + 2\gamma\omega A = \frac{F_0}{m}\sin\phi$

$$2\omega(\gamma - \alpha)A$$

square and add

$$A^2\left(\left[\alpha^2 - 2\gamma\alpha\right] + (\omega_0^2 - \omega^2)\right]^2 + A^2 4\omega^2(\gamma-\alpha)^2 = \left(\frac{F_0}{m}\right)^2$$

$$A(\omega) = \frac{F_0/m}{\sqrt{\left[(\alpha^2 - 2\gamma\alpha) + (\omega_0^2 - \omega^2)\right]^2 + 4\omega^2(\gamma-\alpha)^2}}$$

$$\tan\phi = \frac{2\omega(\gamma-\alpha)}{(\alpha^2 - 2\gamma\alpha) + (\omega_0^2 - \omega^2)}$$