PROBLEMS

4.1 Find the force for each of the following potential energy functions:

(a) $V = cxyz + C$

(b) $V = \alpha x^2 + \beta y^2 + \gamma z^2 + C$

(c) $V = ce^{-(\alpha x + \beta y + \gamma z)}$

(d) $V = cr^n$ in spherical coordinates

4.2 By finding the curl, determine which of the following forces are conservative:

(a) $\mathbf{F} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$

(b) $\mathbf{F} = \mathbf{i}y - \mathbf{j}x + \mathbf{k}z^2$

(c) $\mathbf{F} = \mathbf{i}y + \mathbf{j}x + \mathbf{k}z^3$

(d) $\mathbf{F} = -kr^{-n}\mathbf{e}_r$ in spherical coordinates

4.3 Find the value of the constant c such that each of the following forces is conservative:

(a) $\mathbf{F} = \mathbf{i}xy + \mathbf{j}cx^2 + \mathbf{k}z^3$

(b) $\mathbf{F} = \mathbf{i}(z/y) + c\mathbf{j}(xz/y^2) + \mathbf{k}(x/y)$

4.4 A particle of mass m moving in three dimensions under the potential energy function $V(x, y, z) = \alpha x + \beta y^2 + \gamma z^3$ has speed v_0 when it passes through the origin.

(a) What will its speed be if and when it passes through the point (1, 1, 1)?

(b) If the point (1, 1, 1) is a turning point in the motion ($v = 0$), what is v_0?

(c) What are the component differential equations of motion of the particle?

(*Note:* It is *not* necessary to solve the differential equations of motion in this problem.)

4.5 Consider the two force functions

(a) $\mathbf{F} = \mathbf{i}x + \mathbf{j}y$

(b) $\mathbf{F} = \mathbf{i}y - \mathbf{j}x$

Verify that (a) is conservative and that (b) is nonconservative by showing that the integral $\int \mathbf{F} \cdot d\mathbf{r}$ is independent of the path of integration for (a), but not for (b), by taking two paths in which the starting point is the origin (0, 0), and the endpoint is (1, 1). For one path take the line $x = y$. For the other path take the x-axis out to the point (1, 0) and then the line $x = 1$ up to the point (1, 1).

4.6 Show that the variation of gravity with height can be accounted for approximately by the following potential energy function:

$$V = mgz\left(1 - \frac{z}{r_e}\right)$$

in which r_e is the radius of the Earth. Find the force given by the above potential function. From this find the component differential equations of motion of a projectile under such a force. If the vertical component of the initial velocity is v_{0z}, how high does the projectile go? (Compare with Example 2.3.2.)

4.7 Particles of mud are thrown from the rim of a rolling wheel. If the forward speed of the wheel is v_0, and the radius of the wheel is b, show that the greatest height above

4.1

(a) $V = cxyz + C$

$$F = -\nabla V = -c(yz + xz + xy)$$

(b) $V = \alpha x^2 + \beta y^2 + \gamma z^2 + C$

$$F = -\nabla V = -(2\alpha x + 2\beta y + 2\gamma z)$$

(c) $V(x,y,z) = ce^{-(\alpha x + \beta y + \gamma z)}$

$$F = -\nabla V = -(\alpha + \beta + \gamma)\, c\, e^{-(\alpha x + \beta y + \gamma z)}$$

(d) $V - cr^n$

$$F = -\nabla V = -\frac{\partial}{\partial r} V \hat{r} = ncr^{n-1}\hat{e}_r$$

4.3

(a)

$$\nabla \times F = 0 = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & cx^2 & z^3 \end{vmatrix}$$

$$= \hat{k}(2cx - x) = 0$$

$$c = \frac{1}{2}$$

(b)

$$\nabla \times F = 0 = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{z}{y} & \frac{cxz}{y^2} & \frac{x}{y} \end{vmatrix}$$

$$= i\left(-\frac{x}{y^2} - \frac{cx}{y^2}\right) - j\left(\frac{1}{y} - \frac{1}{y}\right) + k\left(\frac{cz}{y^2} + \frac{z}{y^2}\right)$$

$$c = -1$$