Lecture 2/28/2003

Motion Under a Central Force

Section 8 - Gravitation

Gravitational Force $\vec{F}_g = \dfrac{-m_1 m_2 G}{r^2}\,\hat{e}_r$

Mass of Earth $M_e = 5.976\times10^{24}$ kg

Radius of Earth $R_e = 1.495\times10^{11}$ m

Mass of Sun $M_\odot = 1.99\times10^{30}$ kg

$= 333{,}480\ M_e$

$$G = 6.67\times10^{-11}\ \frac{Nm^2}{kg^2}$$

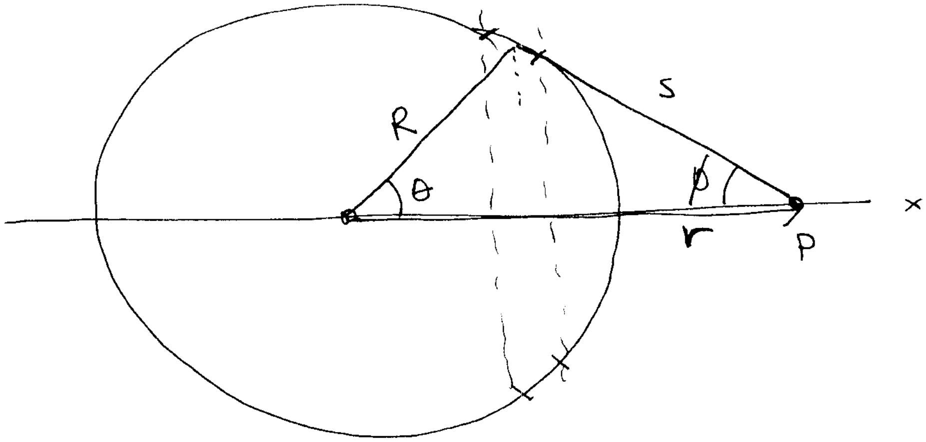
We have been using $\vec{F}_g = \dfrac{-m_1 m_2 G}{r^2}\,\hat{e}_r$ near earth but the earth isn't a point.

$$\vec{F}_g = -m_1 \int_{\text{volume}} \frac{G\,dm}{r^2}\,\hat{e}_r$$

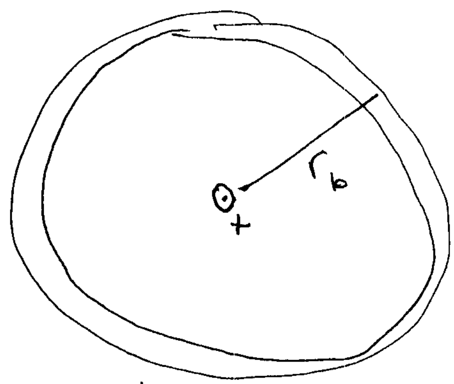
If the mass density is uniform, $dm = \rho\, dV$.

Gravitational Force of Uniform ~~Volume~~ Spherical Shell

surface ~~charge~~ mass density $\sigma = \frac{\text{Mass}}{\text{Area}}$



Cut the sphere into circular bands.



$r_b = R \sin\theta$

The mass of the band

$$\Delta M = \underbrace{R\Delta\theta}_{\text{thickness}} \; \underbrace{2\pi R \sin\theta}_{\text{circumference}} \; \underbrace{\sigma}_{\text{density}} \qquad (6)$$

Law of Cosines

$$s^2 = R^2 + r^2 - 2Rr\cos\theta \qquad (1)$$

$$R^2 = s^2 + r^2 - 2sr\cos\phi \qquad (2) \Longrightarrow \cos\phi = \frac{s^2 + r^2 - R^2}{2sr} \qquad (4)$$

$$2s\,ds = 2Rr\sin\theta\, d\theta \qquad (3)$$

$$\boxed{\frac{s\,ds}{r} = R\sin\theta\, d\theta} \quad (5)$$

The y components of force cancel

$$F_x = \cancel{-m_1 \int \frac{G\sigma\, dS\, d}{S^2} \cos\phi} \qquad -m_1 G \int_{\text{volume}} \frac{dm}{s^2}\cos\phi$$

Use (4) + (5)

$$= -m_1 G \int_{\text{shell}} \frac{(R\,d\theta\, 2\pi R\sin\theta\,\sigma)}{s^2}\left[\frac{s^2+r^2-R^2}{2sr}\right]$$

Use (5)

$$= \frac{-2\pi R\sigma m_1 G}{2r^2}\int_{\text{shell}} \frac{s\,ds}{s^2}\left[\frac{s^2+r^2-R^2}{s}\right]$$

$$= -\frac{2\pi R\sigma m_1 G}{2r^2}\int_{r-R}^{r+R} ds\left[\frac{s^2+r^2-R^2}{s^2}\right]$$

$$= \frac{-\pi R\sigma m_1 G}{r^2}\int_{r-R}^{r+R} ds\left[1 + \frac{r^2-R^2}{s^2}\right]$$

$$= \frac{-\pi R\sigma m_1 G}{r^2}\left[s\Big|_{r-R}^{r+R} - \frac{r^2-R^2}{s}\Big|_{r-R}^{r+R}\right]$$

$$= \frac{-\pi R\sigma m_1 G}{r^2}\Big(2R - (r-R) + (r+R)\Big)$$

$$F_x = -\frac{4\pi R^2 \sigma m_1 G}{r^2} = -\frac{M m_1 G}{r^2}$$

$$M = 4\pi R^2 \sigma$$

Wow!

So the gravitational force of a spherical shell is the same force as if the total mass of the shell was at the origin.

$\Longrightarrow$ We're ok treating the earth as a point mass at its center.

Section 4 — But that was hard

Gravitational Field

$$\vec{g} = \frac{\vec{F}_g}{m} = -\frac{mG}{r^2}\hat{e}_r$$

Gravitational Potential Energy

$$V = -\frac{m_1 m_2 G}{r} = \frac{-m_1 m_2 G}{r} \qquad (V(\infty)=0)$$

Gravitational Potential

$$\Phi = \frac{V}{m_2} = -\frac{mG}{r}$$

Introduce ε_0^g (I just made it up)

$$\vec{g} = -\frac{mG}{r^2}\hat{e}_r = \frac{-m}{4\pi\varepsilon_0^g r^2}\hat{e}_r$$

$$G \equiv \frac{1}{4\pi\varepsilon_0^g} \Rightarrow \varepsilon_0^g = \frac{1}{4\pi G}$$

Everything you learned in UP II works —

"Gauss"
$$\underbrace{\int_s (\vec{g}\cdot\hat{A})\,dA}_{\text{Gravitational flux out of surface.}} = \frac{M_{enc}}{\varepsilon_0^g}$$

Total mass enclosed in surface.

$\Rightarrow$ Gravitational Field inside a spherical shell of mass $m = 0$, since M_{enc} by Gaussian surface $= 0$.

<u>Example</u> Mass density $\rho(r) = \frac{\alpha \rho_0}{r}$

Calculate gravitational field at a distance r.

<u>Sln</u> For spherical symmetry,

$$\int_S (\vec{g} \cdot d\hat{n})\, dA = 4\pi r^2 g(r) = \frac{M_{enc}}{\varepsilon_0^g}$$

where S is a surface of radius r.

$$M_{enc}(r) = \int_0^r (dr\, 4\pi r^2) \frac{\alpha \rho_0}{r} dr$$

$$= 4\pi \alpha \rho_0 \int_0^r r\, dr = 4\pi \alpha \rho_0 \frac{r^2}{2}$$

$$\vec{g}(r) = \frac{M_{enc}}{4\pi r^2 \varepsilon_0^g} \hat{e}_r = \frac{\alpha \rho_0}{2} \frac{4\pi r^2}{4\pi r^2 \varepsilon_0^g} \hat{e}_r$$

$$= \frac{\alpha \rho_0}{2 \varepsilon_0^g} \hat{e}_r$$!! Constant gravitational field

$$= \frac{4\pi G \rho_0}{2} \hat{e}_r$$

Gravitational Energy Density

$$\eta = \frac{1}{2} \varepsilon_0^g \vec{g} \cdot \vec{g}$$

$$= \frac{\vec{g} \cdot \vec{g}}{8\pi G}$$

Example Total Energy of Shell of Mass M

$$\vec{g} = -\frac{MG}{r^2}$$

$$\eta = \frac{M^2 G^2}{r^4} \cdot \frac{-1}{8\pi G} = \frac{M^2 G}{8\pi r^4}$$

$$E = \int_R^\infty \underbrace{4\pi r^2 dr}_{\text{volume of a shell}} \eta = \int_R^\infty \frac{M^2 G}{2r^2} dr = \frac{M^2 G}{2} \left(\frac{-1}{r} \right)_R^\infty$$

$$= \frac{M^2 G}{2R}$$

Gravitational "Capacity"

$$C_g = \frac{M}{\Delta \Phi}$$

For a spherical shell $C_g = 4\pi \varepsilon_0^g R = \frac{R}{G}$

Energy stored in a gravitation capacitor is

$$U = \frac{1}{2} C (\Delta\Phi)^2 = \frac{1}{2} \frac{M^2}{C_g}$$

$$= \frac{1}{2} \frac{M^2}{R/G} = \frac{1}{2} \frac{M^2 G}{R}$$

Lecture 3/3 /2003

Central Forces

Section 3 - Central Forces

Gravity is a central force, the force of gravity is always directed to the center of a spherical mass distribution.

Central Force - $\vec{F} = f(r)\,\hat{e}_r$ - The force is directed towards or away from a fixed center of force at the origin.

General Plan

I. Identify Conserved Stuff.

II. Write EOM, the component EOM.

III. Solve EOM for trajectory.

Possible Conserved Stuff

I. Momentum (NO!)

II. Energy

III. Angular Momentum

<u>Conservation of Energy</u> - If force is conservative,

$\Rightarrow$ if $\nabla \times \vec{F} = 0$. We already showed

for a central force, $\nabla \times \vec{F} = 0$.

$$E = \frac{1}{2} m v^2 + V(r) = \text{constant}$$

$$\vec{F} = -\nabla V(r)$$

$$\boxed{f(r) = -\frac{dV}{dr}}$$

<u>Conservation of Angular Momentum</u> -

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \underbrace{\vec{v} \times (m\vec{v})}_{0} + \vec{r} \times \vec{F}$$

$$= \vec{r} \times f(r)\hat{e}_r = f(r)\, \hat{r} \times \hat{e}_r = 0$$

$\Rightarrow$ Motion occurs in a plane since $\vec{r} \times \vec{v} = \text{constant}$
($\vec{L} \perp$ plane of motion)

$$\vec{L} = \text{constant}$$

$$\frac{|\vec{L}|}{m} \equiv l = \text{constant}$$

<u>what is $\vec{L}$?</u> We have motion around a central point, try plane polar coordinates.

$$\boxed{\begin{aligned} \vec{r}(t) &= r\hat{e}_r \\ \vec{v}(t) &= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \end{aligned}}$$

$$\vec{L} = \vec{r}\times\vec{p} = \vec{r}\times(m\vec{v}) = m\,\vec{r}\times\vec{v}$$

$$\frac{\vec{L}}{m} = (r\hat{e}_r)\times(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta)$$

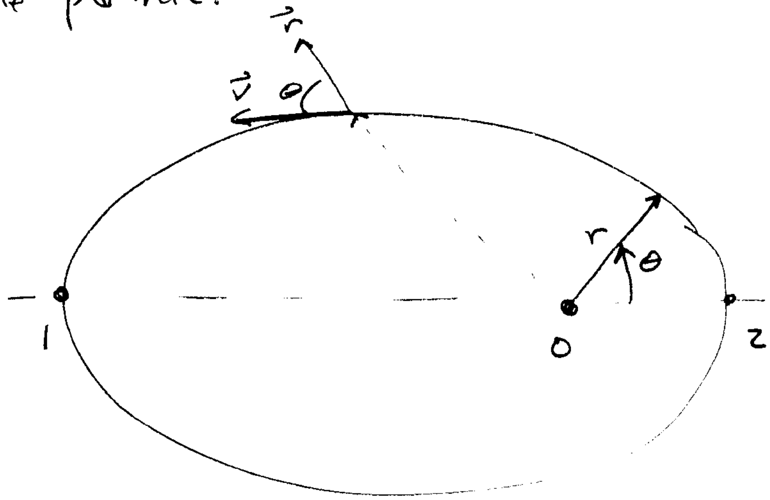
$$= r^2\dot{\theta}\hat{k}$$

$$\frac{|\vec{L}|}{m} = l = r^2\dot{\theta} = \text{constant}$$

What are we looking for?

Trajectory $r(t)$ $\theta(t)$

Orbit $r(\theta)$ - Shape of the path of the particle.



Rate the path is traversed is given by $r(t)$ $\theta(t)$.

At what points is $\dot{r} = 0$ (not moving toward center of force)

$\dot{\theta} = 0$ (none)

At (1) and (2) angular momentum / mass

is $\boxed{rv = l}$

At all other points, $l = |\vec{r} \times \vec{v}| = rv \sin\theta$

Ex - Find trajectory given orbit $r = \theta$

$$l = r^2\dot{\theta} = \text{constant}$$

$$l = \theta^2\dot{\theta} = \theta^2\frac{d\theta}{dt}$$

$$l\,dt = \theta^2 d\theta \qquad \text{Let } \theta = 0 \text{ at } t = 0$$

$$l t = \frac{\theta^3}{3}$$

$$(3lt)^{1/3}$$

$$r(t) = \theta(t) = (3lt)^{1/3}$$

Conservation of Energy

$$\frac{1}{2}mv^2 + V(r) = E = \text{constant}$$

$$v^2 = \vec{v}\cdot\vec{v} = \dot{r}^2 + r^2\dot{\theta}^2$$

$$\frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r) = E$$

Something that works

$$r = \frac{1}{u}$$

$$\dot{r} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\dot{\theta}\frac{du}{d\theta}$$

$$\frac{m}{2}\left(\frac{1}{u^4}\dot{\theta}^2\left(\frac{du}{d\theta}\right)^2 + \frac{1}{u^2}\dot{\theta}^2\right) + V(u^{-1}) = E$$

$$\dot{\theta}^2 = \frac{l}{r^2} = l u^2$$

$$\frac{1}{2}m\left(l^2\left(\frac{du}{d\theta}\right)^2 + l^2 u^2\right) + V(u^{-1}) = E$$

Energy Eqn of Orbit

$$\frac{1}{2}ml^2\left(\left(\frac{du}{d\theta}\right)^2 + \cancel{l}u^2\right) + V(u^{-1}) = E$$

Example What central force generates trajectory $r = \theta$

$$u = \frac{1}{\theta} \qquad \frac{du}{d\theta} = -\frac{1}{\theta^2} = -u^2$$

Substitute $\frac{1}{2}ml^2\left(u^4 + u^2\right) + V(u^{-1}) = E$

$$V(u^{-1}) = E - \frac{1}{2} m l^2 \left(u^4 + u^2 \right)$$

$$V(r) = E - \frac{1}{2} m l^2 \left(\frac{1}{r^4} + \frac{1}{r^2} \right)$$

$$f(r) = -\frac{dV}{dr} = -\frac{1}{2} m l^2 \left(\frac{4}{r^5} + \frac{2}{r^3} \right)$$

$$= -m l^2 \left(\frac{2}{r^5} + \frac{1}{r^3} \right)$$