Section - Angular Momentum Results

System N particles with mass m_i at position $\vec{r}_i$ with velocity $\vec{v}_i$

<u>Angular Momentum</u> $\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i \vec{r}_i \times m_i \vec{v}_i$

Torque $(\vec{N})$

$$\vec{N} = \frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i$$

external forces only.

<u>Conservation of Angular Momentum</u> If $\vec{F}_i = 0$, the system is isolated $\vec{N} = 0$ and

$$\dot{\vec{L}} = 0$$

$$\Rightarrow \vec{L} = \text{constant}$$

Angular momentum is conserved.

Section - Rotation about Center of Mass

Particles position and velocity with respect to center of mass. (Primed vectors)

$$\vec{r}_i' + \vec{r}_{cm} = \vec{r}_i$$

$$\vec{v}_i' + \vec{v}_{cm} = \vec{v}_i$$

$$\vec{L} = \sum_i (\vec{r}_i' + \vec{r}_{cm}) \times m_i (\vec{v}_i' + \vec{v}_{cm})$$

$$= \underbrace{m \vec{r}_{cm} \times \vec{v}_{cm}}_{\text{Angular momentum of center of mass about origin}} + \underbrace{\sum \vec{r}_i' \times m_i \vec{v}_i'}_{\text{Angular momentum of system about center of mass.}}$$

Likewise Kinetic Enery

$$T = \sum_i \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i$$

$$= \frac{1}{2} m v_{cm}^2 + \sum_i \frac{1}{2} m_i v_i'^2$$

Section - Reduced Mass

Consider an isolated two particle system with Center of Mass at origin.

Defn Center of Mass

$$m_1 \vec{r}_1{}' + m_2 \vec{r}_2{}' = 0$$

$$\vec{r}_2{}' = -\frac{m_1}{m_2} \vec{r}_1{}'$$

EOM Particle 1

$$m_1 \frac{d^2 \vec{r}_1{}'}{dt^2} = \vec{F}_{21}$$

Displacement Vector from 2 to 1

$$\vec{r}_{21} = \vec{r}_1{}' - \vec{r}_2{}' = \vec{r}_1{}' + \frac{m_1}{m_2} \vec{r}_1{}'$$

$$\vec{r}_1{}' = \frac{1}{1 + \frac{m_1}{m_2}} \vec{r}_{21}$$

Substitute into EOM

$$\frac{m_1}{1 + m_1/m_2} \frac{d^2 \vec{r}_{21}}{dt^2} = \vec{F}_{21}$$

Reduced mass

$$\mu = \frac{m_1}{1 + m_1/m_2} = \frac{m_1 m_2}{m_1 + m_2}$$

$$\mu \frac{d^2 \vec{r}_{21}}{d\tau^2} = \vec{F}_{21}$$

$\Rightarrow$ Dynamics of two particle system is the same as a one particle system where the INERTIAL mass ~~replace~~ is replaced with the reduced mass.

If $\vec{F}_{21}$ is gravity, $\vec{F}_{21} = f(r_{21})\hat{e}_r$

$$\mu \frac{d^2 \vec{r}_{21}}{dt^2} = -\frac{m_1 m_2 G}{r^2} \hat{e}_r$$

not replaced by μ

Return to any formula from chapter 6

$$a = \frac{m l^2}{K} = \frac{\mu l^2}{m M G}$$

$$\tau^2 = \frac{4\pi^2 m}{K} a^3$$

$$= \frac{4\pi^2 \mu}{m_1 m_2 G} a^3$$

$$= \frac{4\pi^2}{(m_1+m_2) G} a^3$$

Section Rockets

Consider a system ~~that~~ whose mass changes as $m(t)$ by transferring mass from the enviroment, to the system

Let $\vec{p}_{sys}$ be the momentum of the system.

Let $\vec{p}_{env}$ be " environment.

Assume no forces act on the mass transferred to environment (just irrelevant terms we don't have to take care of).

$$\vec{F}_{ext} = \frac{d}{dt}\left(\vec{p}_{sys} + \vec{p}_{env}\right)$$

$$\vec{p}_{sys} = m(t)\,\vec{v}_{cm}(t)$$

$$\frac{d\vec{p}_{sys}}{dt} = \dot{m}\,\vec{v}_{cm} + m\,\dot{\vec{v}}_{cm}$$

negative because mass is taken out of environment.

$$\Delta\vec{p}_{env} = -\Delta m\,\vec{v}_{mass}$$

velocity of the transferred mass.

$\vec{v}_{rel} \equiv$ relative velocity ~~bet~~ of the mass Δm and the system. (The nozzle velocity in a rocket). ($\vec{V}$ in text)

$$\vec{v}_{mass} = \vec{v}_{cm} + \vec{v}_{rel}$$

$$\frac{d\vec{p}_{env}}{dt} = -\dot{m}\left(\vec{v}_{cm} + \vec{v}_{rel}\right)$$

<u>Put it all together</u>

$$\vec{F}_{ext} = \frac{d\vec{p}_{sys}}{dt} + \frac{d\vec{p}_{env}}{dt}$$

$$= m\dot{\vec{v}}_{cm} + \dot{m}\vec{v}_{cm} - \dot{m}\left(\vec{v}_{cm} + \vec{v}_{rel}\right)$$

$$\cancel{= \dot{m}\vec{v}_{cm}}$$

$$= m\dot{\vec{v}}_{cm} - \dot{m}\vec{v}_{rel}$$

<u>Thrust</u> $\dot{m}\vec{v}_{rel}$ Another thing that acts like a force

$$\vec{F}_{ext} + \dot{m}\vec{v}_{rel} = m\dot{\vec{v}}_{cm}$$

Suppose $\vec{F}_{ext} = 0$,

$$\frac{dm}{dT}\vec{v}_{rel} = \dot{m}\,\vec{v}_{rel} = m\dot{\vec{v}}_{cm} = m\frac{d\vec{v}_{cm}}{dT}$$

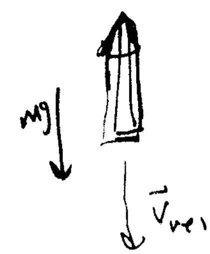
$$\vec{v}_{rel}\,dm = m\,d\vec{v}_{cm}$$

$$\int_{m_0}^{m}\vec{v}_{rel}\frac{dm}{m} = \int_{\vec{v}_0}^{\vec{v}} d\vec{v}_{cm}$$

$$\vec{v}_{rel}\ln\frac{m}{m_0} = \vec{v}_{cm} - \vec{v}_0$$

$$\vec{v}_{cm} = \vec{v}_0 + \vec{v}_{rel}\ln\frac{m}{m_0}$$

Suppose we have a rocket fired near earth



$$\vec{F}_{ext} = -mg\hat{k}$$

$$\vec{F}_{ext} + \dot{m}\vec{v}_{rel} = m\dot{\vec{v}}_{cm}$$

Let $\underbrace{\dot{m} = -\gamma}_{\text{mass of rocket decreasing}}$ $\quad \vec{v}_{rel} = -v_e\hat{k}$

$$-m(t)g\hat{k} + \gamma v_e\hat{k} = m(t)\frac{d\vec{v}_{cm}}{dt}$$

$$m(t) = m_0 - \gamma t \qquad \vec{v}_{cm} = v\hat{k}$$

$$(m_0 - \gamma t)g + \gamma v_e = (m_0 - \gamma t)\frac{dv}{dt}$$

$$\int_0^v dt\left[-g + \frac{\gamma v_e}{m_0 - \gamma t}\right] = \int_0^v dv$$

$$-gt + \gamma v_e\int_0^t \frac{dt}{m_0 - \gamma t} = v$$

$$u = m_0 - \gamma t \qquad du = -\gamma\, dt$$

$$-gt - v_e \int_{u_0}^{u} \frac{du}{u} = v$$

$$-gt - v_e \ln \frac{u}{u_0} = v$$

$$-gt - v_e \ln \frac{m_0 - \gamma t}{m_0} = v$$

$$v(t) = -gt - v_e \ln\left(1 - \frac{\gamma t}{m_0}\right)$$

Lecture - <u>Center-of-Mass</u>

Section - Collisions in Center of Mass

> What happened to all the coefficient of restitution jazz?

$$T = \frac{1}{2} M v_{cm}^2 + \underbrace{\sum \frac{1}{2} m_i \vec{v}_i'^2}_{\text{convertable energy}}$$

$\Rightarrow$ Suggests the Center of Mass ~~System~~ Frame may be useful for analyzing collisions.

<u>Center of Mass System</u> - The coordinate system where center of mass is stationary.

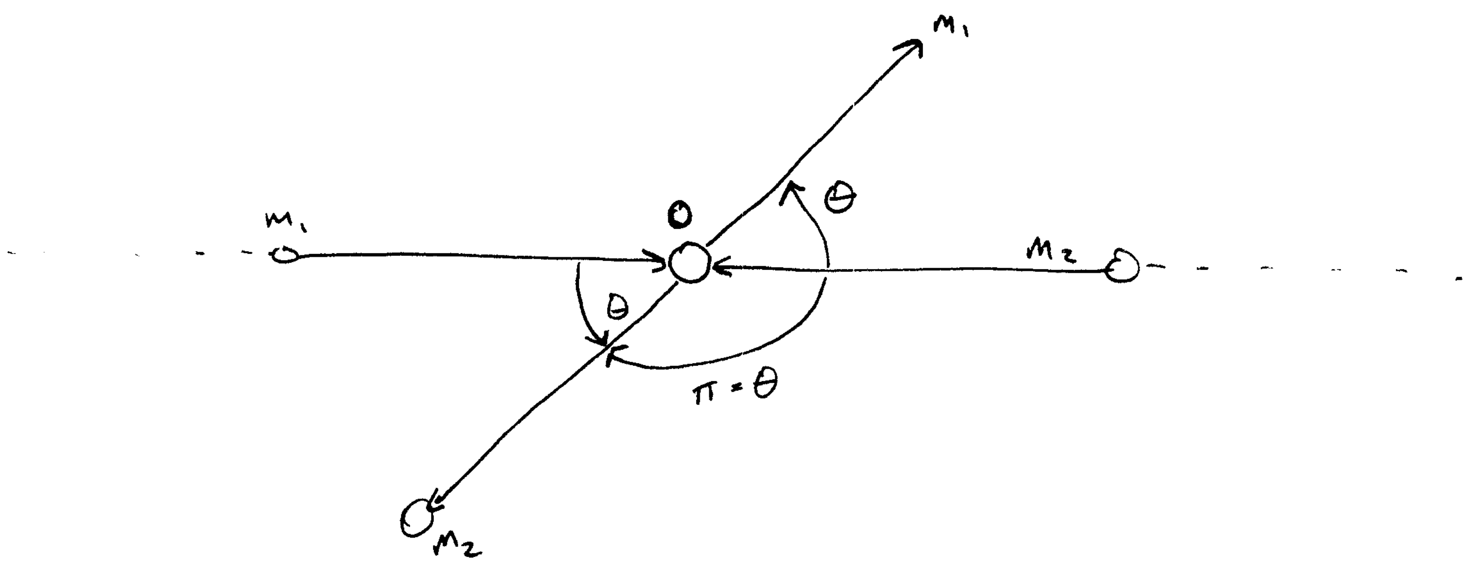
$$\vec{v}_i = \vec{V}_{cm} + \vec{v}_i'$$

<u>Total Momentum in CM</u> — $\vec{P}' = M\vec{V}_{cm} = 0$

$$\vec{P}' = \vec{P}_1^{I'} + \vec{P}_2^{I'} = \vec{P}_1^{F'} + \vec{P}_2^{F'} = 0$$

$$\Rightarrow \vec{P}_1^{F'} = -\vec{P}_2^{F'}$$

$\Rightarrow$ Particles trajectory lies on a line.



IN Center of Mass

x-component momentum

Positive

$$m_1 v_1^{I\prime} + m_2 v_2^{I\prime} = m_1 v_1^{F\prime} \cos\theta - m_2 v_2^{F\prime} \cos\theta$$

y-component

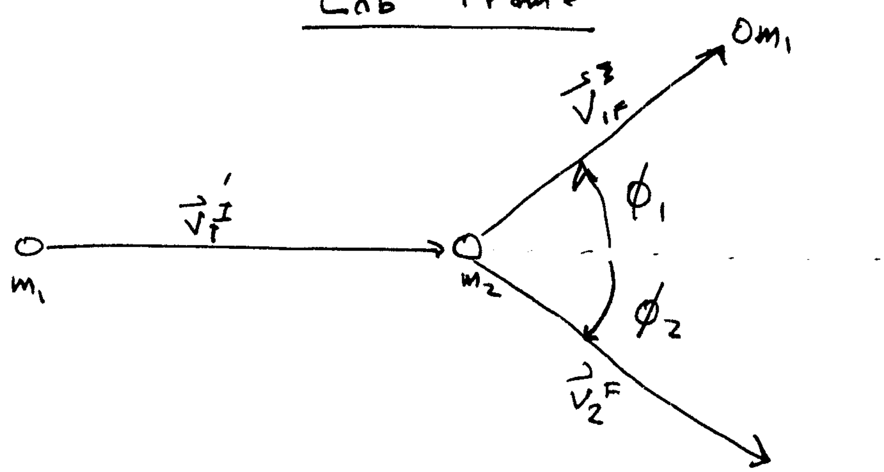
$$0 = m_1 v_1^{F\prime} \sin\theta - m_2 v_2^{F\prime} \sin\theta$$

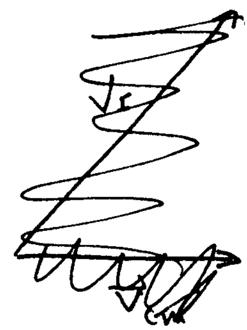
energy

$$\frac{1}{2} m_1 v_1^{I\prime 2} + \frac{1}{2} m_2 v_2^{\prime I 2} = \frac{1}{2} m_1 v_1^{\prime F 2} + \frac{1}{2} m_2 v_2^{\prime F 2} + Q$$

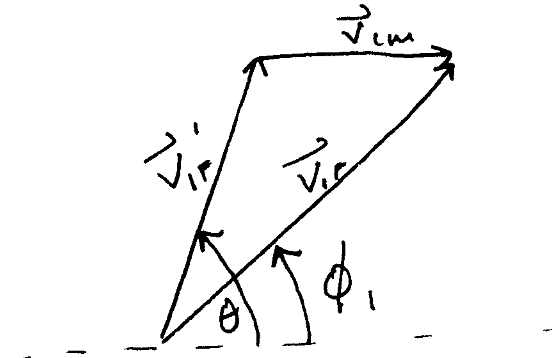
Relation between Lab and CM Frames — Transformation defined by $\vec{v}_i = \vec{V}_{cm} + \vec{v}_i^{\,\prime}$

Lab Frame





$$\vec{V}_{1F} = \vec{V}_{cm} + \vec{V}_1^{F'}$$



x-component

$$v_{1}^{F} \cos\phi_1 = v_{cm} + v_{1F}' \cos\theta$$

x-component

$$v_{1}^{F} \sin\phi_1 = v_{1F}^{F'} \sin\theta$$

$$\tan\phi_1 = \frac{v_1^{F'} \sin\theta}{v_{cm} + v_{1F}' \cos\theta} = \frac{\sin\theta}{\gamma + \cos\theta}$$

$$\gamma = \frac{v_{cm}}{v_{1F}'}$$

After a page of algebra, for elastic collisions,

$$\gamma = \frac{m_1}{m_2}$$

<u>Example</u> If you completed the bowling ball problem from last lecture, you found the ball scattered at an angle of $10°. = \phi_1$

What was the scattering angle in CM?

<u>Sln</u> $$\tan\phi_1 = \frac{\sin\theta}{\gamma + \cos\theta}$$

$$\gamma = \frac{m_1}{m_2} = 5$$

$$\tan\phi_1\left(5+\cos\theta\right) = \sin\theta$$

$$\tan^2\phi_1\left(25+10\cos\theta+\cos^2\theta\right) = \sin^2\theta = 1-\cos^2\theta$$

$$\left(1+\tan^2\phi_1\right)\cos^2\theta + 10\tan^2\phi_1\cos\theta + 25\tan^2\phi_1 - 1 = 0$$

$$1.03\cos^2\theta + 0.31\cos\theta - 0.22 = 0$$

$$\cos\theta = \frac{-0.31 \pm \sqrt{(0.31)^2 + 4(1.03)(0.22)}}{2(1.03)}$$

$$= 0.3356$$

$$\theta = 70^\circ$$

<u>What is velocity of CM?</u> Before collision?

$$\vec{V}_{cm} = \frac{1}{m_b+m_p} \quad m_b\vec{V}_b^{\,I} + m_p\vec{V}_p^{\,I}$$

$$= \frac{5}{6}V_0$$

What about after collision?