Section - Systems of Particles

Consider a system of N (possibly interacting) particles with mass m_i, position $\vec{r}_i$, and velocity $\dot{\vec{r}}_i = \vec{v}_i$. Now we can describe anything.

$\boxed{\text{Choice of system arbitrary.}}$

Total Mass of System (m) $\quad m = \sum_i m_i$

Center of Mass $(\vec{r}_{cm})$ $\quad \vec{r}_{cm} = \dfrac{\sum m_i \vec{r}_i}{m}$

Total Momentum of System $(\vec{p})$ $\quad \vec{p} = \sum \vec{p}_i = \sum m_i \vec{v}_i$

Velocity of Center of Mass $(\vec{v}_{cm})$

$$\vec{v}_{cm} = \dot{\vec{r}}_{cm} = \frac{1}{m} \sum m_i \dot{\vec{r}}_i$$

$$m\vec{v}_{cm} = \sum m_i \dot{\vec{r}}_i = \sum m_i \vec{v}_i = \vec{p}$$

$\boxed{\text{The total momentum of the system is the same as a point mass, with total mass of the system moving at the velocity of the center of mass.}}$

Total Force on Particle i

Net External Forces $\vec{F}_i$

Internal Forces $\sum_j \vec{F}_{ji}$

Newton II -

$$\dot{\vec{P}}_i = \vec{F}_i + \sum_j \vec{F}_{ji}$$

Sum Over all Particles

$$\sum_i \dot{\vec{P}}_i = \sum_i \vec{F}_i + \sum_i \sum_j \vec{F}_{ji}$$

Newton III - The internal force come in pairs $\vec{F}_{ji} = -\vec{F}_{ij}$, so the second sum is zero.

$$\sum_i \dot{\vec{P}}_i = \sum_i \vec{F}_i$$

Time Rate of Change of Total Momentum

$$\dot{\vec{P}} = m\dot{\vec{V}}_{cm} = m\vec{a}_{cm} = \sum \dot{\vec{P}}_i = \sum \vec{F}_i$$

The acceleration of the center of mass is the same as the acceleration of a point mass, with the total mass of the system, acted on by the sum of the forces acting on the system.

<u>Body Forces</u> Force of Gravity

$$\vec{F}_i = m_i \vec{g}$$

$$\sum \vec{F}_i = \sum m_i \vec{g} = m\vec{g} = m\vec{a}_{cm}$$

If no <u>external</u> forces,

$$\sum \vec{F}_i = 0 = \dot{\vec{P}} \implies \vec{P} = \text{constant}$$

<u>Total Momentum is Conserved.</u>

Section - Angular Momentum

Angular Momentum ($\vec{L}$) about the origin.

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i \left(\vec{r}_i \times m\vec{v}_i\right)$$

$$\frac{d\vec{L}}{dt} = \sum_i \left(\dot{\vec{r}}_i \times m\vec{v}_i + \vec{r}_i \times m\dot{\vec{v}}_i\right)$$

$$\vec{v}_i \times \vec{v}_i = 0$$

$$\dot{\vec{v}}_i = \vec{a}_i$$

$$m\vec{a}_i = \vec{F}_i + \sum_j \vec{F}_{ji}$$

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \left(\vec{F}_i + \sum_j \vec{F}_{ji}\right)$$

$$= \sum_i \vec{r}_i \times \vec{F}_i + \sum_i \sum_j \vec{r}_i \times \vec{F}_{ji}$$

Newton III again — In the sum, we will have pairs ~~$\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \vec{F}_{ji}$~~

$$\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij}$$

$$\vec{F}_{ji} = -\vec{F}_{ij}$$

$$(\vec{r}_j - \vec{r}_i) \times \vec{F}_{ij}$$

The displacement vector from particle i to particle j is $\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$

If $\vec{F}_{ij}$ is central, $\vec{F}_{ij} \parallel \vec{r}_{ij}$ and the cross product is zero.

$$\boxed{\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i \equiv \vec{N} \quad \text{Torque}}$$

If system is isolated $\dot{\vec{L}} = 0$, $\vec{L} = \text{constant}$ angular momentum is conserved.

The conservation of momentum, angular momentum, and energy works in all known forces

<u>Section</u> <u>Rotation about center of mass</u>

Write particles position and velocity about the center of mass $\vec{r}_i{}' + \vec{r}_{cm} = \vec{r}_i$ $\vec{v}_i{}' + \vec{v}_{cm} = \vec{v}_i$

$$L = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i (\vec{r}_i{}' + \vec{r}_{cm}) \times m_i(\vec{v}_i{}' + \vec{v}_{cm})$$

$$= \left(\sum_i m_i\right) \vec{r}_{cm} \times \vec{v}_{cm} + \vec{r}_{cm} \times \sum_i m_i \vec{v}_i{}'$$

$$+ \left(\sum m_i \vec{r}_i{}'\right) \times \vec{v}_{cm} + \sum \vec{r}_i{}' \times m \vec{v}_i{}'$$

$$\sum m_i \vec{r}_i{}' + m_i \vec{r}_{cm} = \sum m_i \vec{r}_i$$

$$\cancel{m\vec{r}_{cm}}$$

$$\underbrace{\sum m_i \vec{r}_i{}'}_{0} + m \vec{r}_{cm} = m \vec{r}_{cm}$$

Likewise $\sum m_i \vec{v}_i{}' = 0$

$$\vec{L} = \underbrace{m\,\vec{r}_{cm} \times \vec{v}_{cm}}_{\substack{\text{Angular Momentum} \\ \text{of center of mass} \\ \text{about origin.}}} + \underbrace{\sum \vec{r}_i{}' \times m_i \vec{v}_i{}'}_{\substack{\text{Angular Momentum} \\ \text{of system about} \\ \text{center of mass.}}}$$

<u>Kinetic Energy</u> (Likewise)

$$T = \sum_i \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i$$

$$= \frac{1}{2} m v_{cm}^2 + \sum_i \frac{1}{2} m_i v_i^{2\prime}$$

Section - Reduced Mass

Consider an isolated two particle system with CM at origin.

$$m_1 \vec{r}_1{}' + m_2 \vec{r}_2{}' = 0$$

displacement from CM

$$\cancel{\vec{r}_1{}' = -\frac{m_2}{m_1}\vec{r}_2{}'} \qquad \vec{r}_2{}' = -\frac{m_1}{m_2}\vec{r}_1{}'$$

EOM of particle 1

$$m_1 \frac{d^2 \vec{r}_1{}'}{dt^2} = \vec{F}_{21} = f(r_{21})\,\hat{r}_{21}$$

~~Suppose $\vec{F}_1$ is an:~~

Displacement Vector

$$\vec{r}_{21} = \vec{r}_1{}' - \vec{r}_2{}'$$

$$= \vec{r}_1{}'\left(1 + \frac{m_1}{m_2}\right)$$

$$\frac{\vec{r}_{21}}{1 + \frac{m_1}{m_2}} = \vec{r}_1{}'$$

$$\frac{m_1}{1 + \frac{m_1}{m_2}} \frac{d^2 \vec{r}_{21}}{dt^2} = f(r_{21})\,\hat{r}_{21}$$

Define reduced mass

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$\mu \frac{d^2 \vec{r}_{21}}{dt^2} = f(r_{21}) \hat{r}_{21}$$

1 orbits 2

The effect of having two particles instead of a fixed center of force is to replace the inertial mass m_1 with μ.

If f is gravity, $f = -\frac{m_1 m_2 G}{r^2}$

not $\frac{\mu m_2 G}{r^2}$

So we can go back and fix all our orbit equations. For example

$$\tau^2 = \frac{4\pi^2}{GM_\odot} a^3 \Rightarrow \tau^2 = \frac{4\pi^2}{G(M_\odot + m)} a^3$$

Section 1
~~Lecture 3/29~~ – Collisions

In a collision between N particles, the forces are very short range and generally unknown. We observe the masses m_i and the momenta $\vec{P}_i$ before and after collision. In nuclear collision, the m_i may change

Momentum Conservation

$$\underbrace{\sum \vec{P}_i^{I}}_{\text{initial momentums}} = \sum \vec{P}_i^{F}$$

final momentums

Energy Conservation

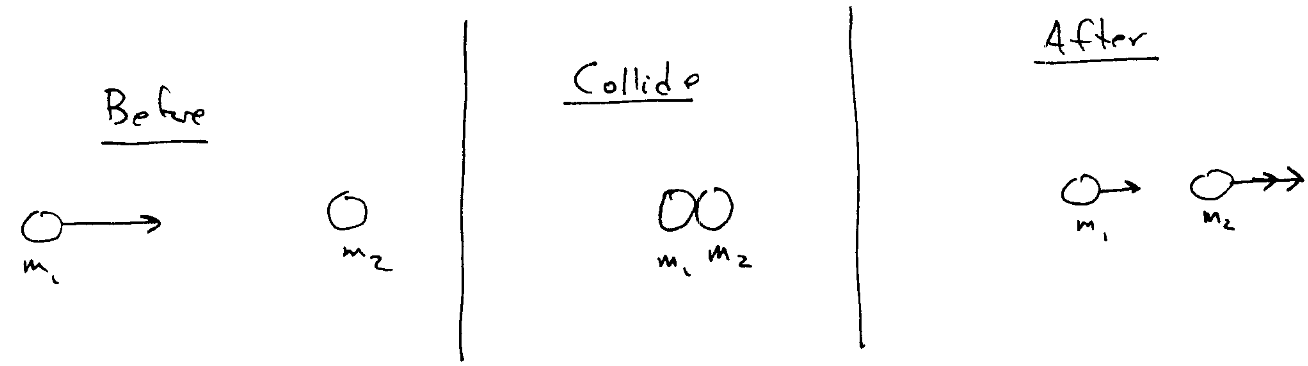
$$\sum_i \frac{(P_i^I)^2}{2m_i} = \sum_i \frac{(P_i^F)^2}{2m_i} + Q$$

$Q \equiv$ Energy lost to envirment in collision. If $Q > 0$ the particles have less energy after the collision.

Section 2 - Direct Collisions

Back to UP III.

Suppose the particles before and after the collision travel on a single line. $\Longrightarrow$ Direct Collision (Head on collision)



Since all in a line, we don't have to use vectors anymore.
They all have the same unit vector.

~~To further simplify, work in a coordinate system where m_2 is at rest, and only two particles collide.~~

Momentum Conservation (p have signs)

$$p_2^I + p_1^I = p_1^F + p_2^F$$

Energy Conservation

$$\frac{(P_2^I)^2}{2m_2} \quad \frac{(P_1^I)^2}{2m_1} = \frac{(P_1^F)^2}{2m_1} + \frac{(P_2^F)^2}{2m_2} + Q$$

Relative Velocity

$V^I = |v_2^I - v_1^I|$ Before Collision

$V^F = |v_2^F - v_1^F|$ After Collision

Coefficient of Restitution

(How much velocity is restored after collision?)

$$\varepsilon = \frac{V^F}{V^I}$$

Elastic Collision – No energy lost or gained from environment, $Q = 0$ and $\varepsilon = 1$.

Pf $\quad m_1 v_1^I + m_2 v_2^I = m_1 v_1^F + m_2 v_2^F$

If elastic, $\varepsilon = 1 \quad v_2^F - v_1^F = v_1^I - v_2^I$

$(m_1 v_1^I + m_2 v_2^I)(v_1^I - v_2^I) = (m_1 v_1^F + m_2 v_2^F)(v_2^F - v_1^F)$

$m_1 (v_1^I)^2 - m_2 (v_2^I)^2 + (m_1 + m_2)(v_1^I v_2^I)$

$= -m_1 v$

General Final Velocities

$$v_1^F = \frac{(m_1 - \varepsilon m_2) v_1^I + (m_2 + \varepsilon m_2) v_2^I}{m_1 + m_2}$$

$$v_2^F = \frac{(m_1 + \varepsilon m_2) v_1^I + (m_2 - \varepsilon m_1) v_2^I}{m_1 + m_2}$$

Totally Inelastic $\quad \varepsilon = 0$

$$v_1^F = \frac{m_1 v_1^I + m_2 v_2^I}{m_1 + v_2} = v_{cm}$$

$$= v_2^F$$

The particles move with the same velocity.

Totally Elastic $\quad \underline{\varepsilon = 1}$ AND $\underline{m_1 = m_2}$

$$v_1^F = v_2$$

$$v_2^F = v_1$$

Exchange Velocites.

Energy Loss

$$Q = \frac{1}{2}\mu(v^I)^2(1-\varepsilon^2)$$

Left as homework problem.

$\Longrightarrow$ How large can Q be?

$$Q_{max} = \frac{1}{2}\mu(v^I)^2$$

Recall: The kinetic energy with respect to center of mass can be separated as

$$T = \frac{1}{2}Mv_{cm}^2 + \sum_i \frac{1}{2}m_i v_i'^2$$

velocity with respect to cm.

With about a page of algebra, you can show

$$Q_{max} = \sum \frac{1}{2}m_i v_i'^2$$

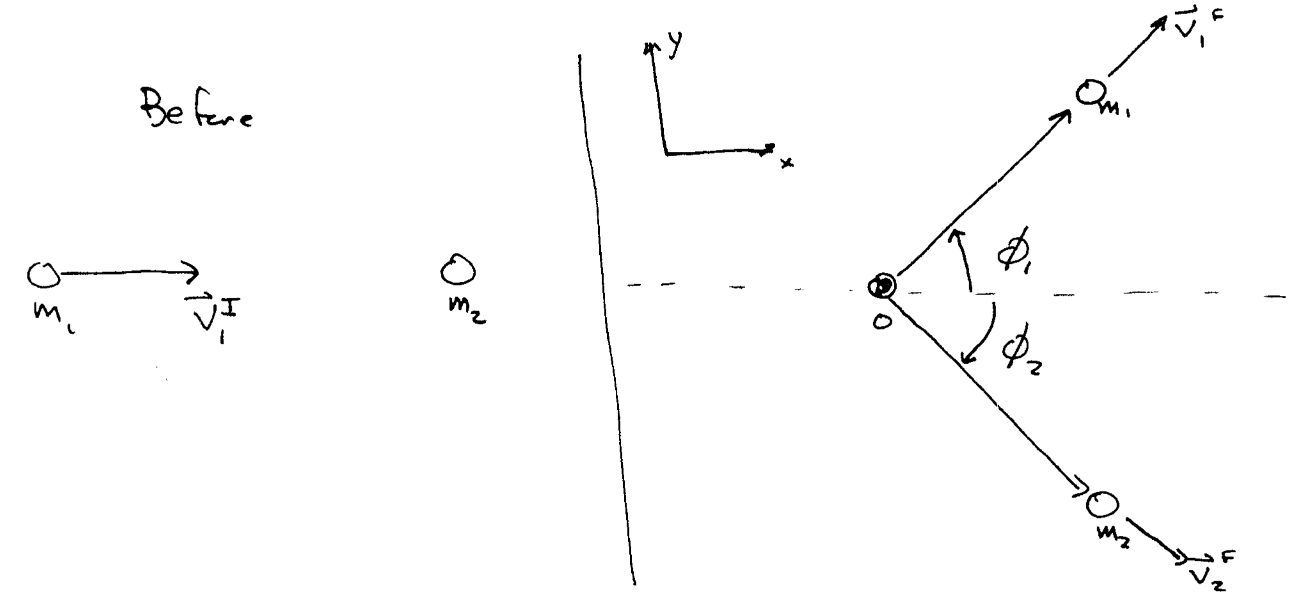
$$\boxed{\text{The energy } \cancel{\text{of}} \text{ the motion of the center of mass cannot be converted to another form. Required to}}$$ conserve momentum,

$\Longrightarrow$ Q does not depend on coordinate system,
since depends relative velocity.

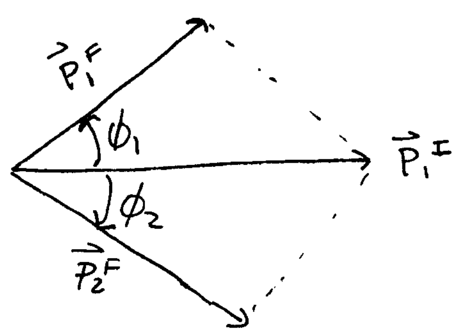
Section

Collisions in Two Dimensions

Select a coordinate system, the lab system, where m_2 is at rest before collision.



Momentum Conservation



x-component

$$p_1^I = p_1^F \cos\phi_1 + p_2^F \cos\phi_2$$

y-component

$$0 = p_1^F \sin\phi_1 - p_2^F \sin\phi_2$$

Energy

$$\frac{(p_1^I)^2}{2m_1} = \frac{(p_1^F)^2}{2m_1} + \frac{(p_2^F)^2}{2m_2} + Q$$

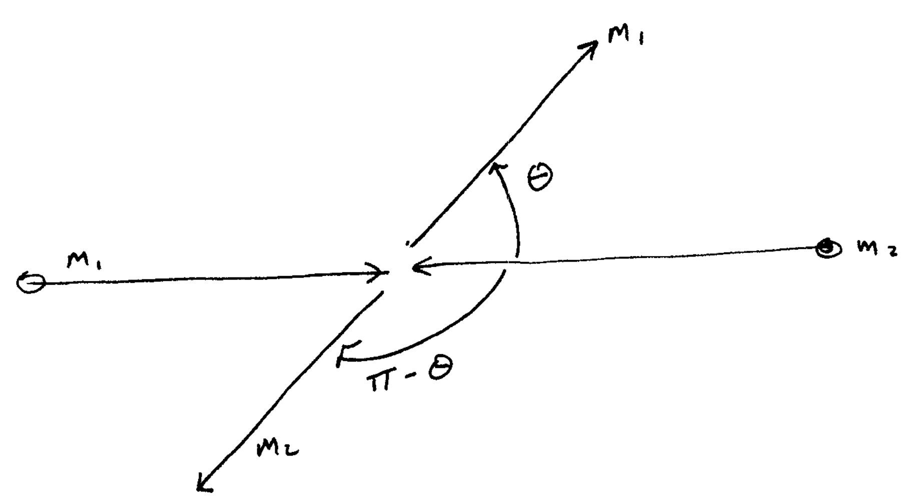
$\boxed{\text{How does all the coefficient of restitution jazz work here?}}$

Consider the collision in CM coordinate system, center of mass fixed at origin.

$$\vec{v}_1 = \vec{v}_{cm} + \vec{v}_1{}' \qquad \vec{v}_2 = \vec{v}_{cm} + \vec{v}_2{}'$$

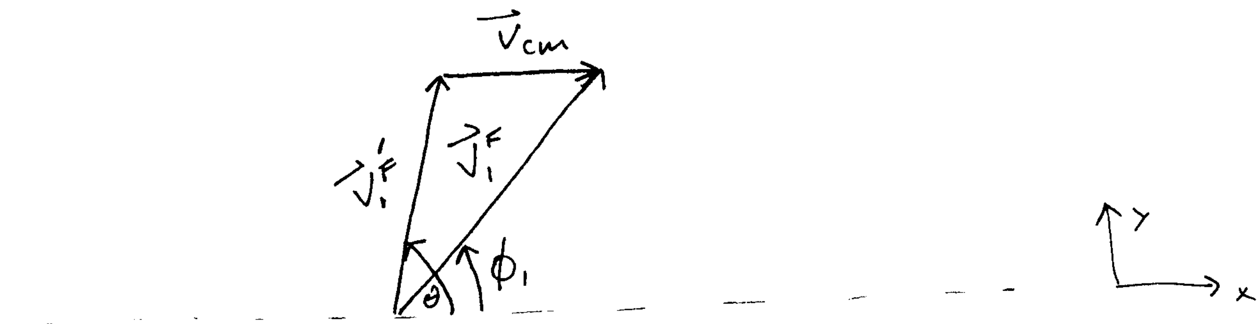
Total Momentum $\quad \vec{P} = M\vec{v}_{cm} \quad \Rightarrow \quad \vec{P} = 0$

$$\vec{p}_1^{I\prime} + \vec{p}_2^{I\prime} = 0 = \vec{p}_1^{F\prime} + \vec{p}_2^{F\prime}$$



Particles move in line after collision. Can define relative velocity v^I and v^F, coefficient of restitution. It all still works in CM, ONLY! Note whether particles exchange velocity depends on scattering angle.

Relation between lab and CM scattering angles.



x component

$$v_{1F} \cos\phi_1 = v_{cm} + v_1^{\prime F} \cos\theta$$

x-component

$$v_{1F} \sin\phi_1 = v_{1F}' \sin\theta$$

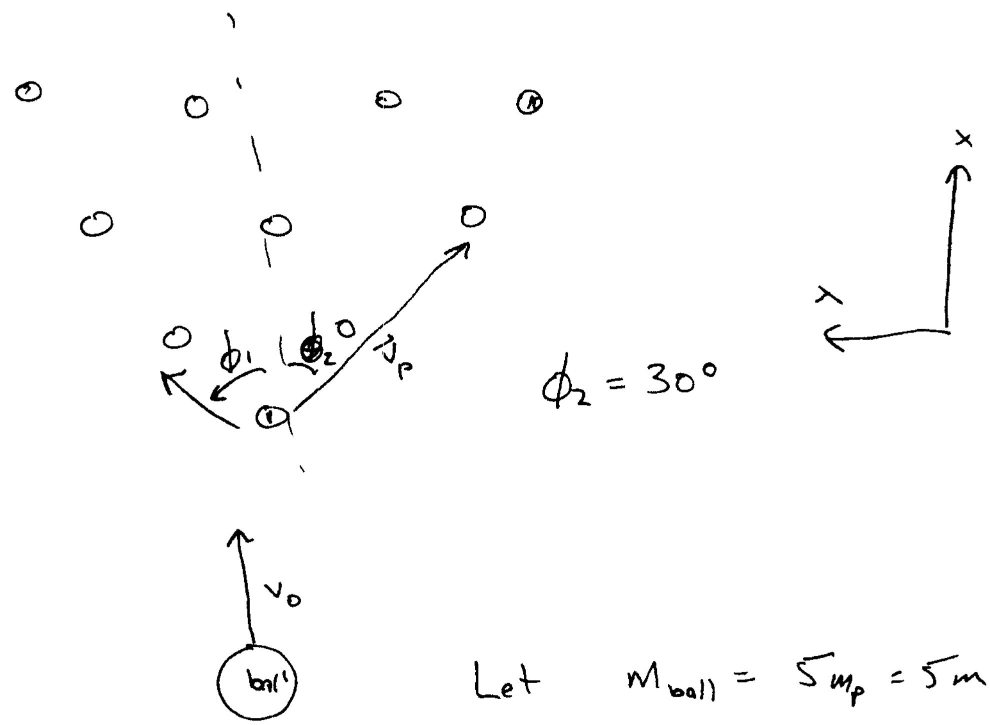
$$\tan\phi_1 = \frac{v_{1F}' \sin\theta}{v_{cm} + v_1^{\prime F}\cos\theta} = \frac{\sin\theta}{\gamma + \cos\theta}$$

$$\gamma = v_{cm} / v_{1F}'$$

For an elastic collision, $\gamma = m_1/m_2$

Through a bit of algebra.

Section - Example 1/10 Split



$\phi_2 = 30°$

Let $M_{ball} = 5m_p = 5m$

$$\vec{P}^{I}_{ball} = \vec{P}^{F}_{ball} + \vec{P}^{F}_{pin}$$

$$\frac{1}{2} m_{ball} V^{I^2}_{ball} = \frac{1}{2} M_{ball} V^{F^2}_{ball} + \frac{1}{2} M_{pin} V^{F^2}_{pin} + Q$$

Let $Q = 0$, approximate collision as elastic, no rotation.

<u>x-component momentum</u>

$$5mv_0 = 5mv_b^F \cos\phi_1 + mv_p^F \cos 30° \qquad (1)$$

<u>y-component momentum</u>

$$0 = 5mv_b^F \sin\phi_1 - mv_p^F \sin 30° \qquad (2)$$

Energy $\quad 5v_o^2 = 5v_b^{F^2} + v_p^{F^2}$

$$v_o^2 - \frac{1}{5}v_p^{F^2} = v_b^{F^2} \qquad (3)$$

Manipulate

$$v_o - \frac{1}{5}v_p^F \cos 30^\circ = v_b^F \cos\phi_1 \qquad (4)$$

$$\frac{1}{5}v_p^F \sin 30^\circ = v_b^F \sin\phi_1 \qquad (5)$$

$(4)^2 + (5)^2 = (6)$

$$v_o^2 - \frac{2}{5}v_p^F v_o \cos 30^\circ + \frac{1}{25}v_p^{F^2}\cos^2 30^\circ + \frac{1}{25}v_p^{F^2}\sin^2 30^\circ = v_b^{F^2} \qquad (6)$$

(6) − Energy

$$-\frac{2}{5}v_p^F v_o \cos 30^\circ + \frac{1}{25}v_p^{F^2} + \frac{1}{5}v_p^{F^2} = 0$$

$$-10 v_o \cos 30^\circ + 6 v_p^F = 0$$

$$v_p^F = \frac{10}{6} v_o \cos 30^\circ = 1.44 v_o$$

Angle (5)/(4)

$$\tan\phi_1 = \frac{\frac{1}{5} v_p^F \sin 30^\circ}{v_0 - \frac{1}{5} v_p^F \cos 30^\circ}$$

$$= \frac{1}{\frac{5 v_0}{v_p^F \sin 30^\circ} - \frac{\cos 30^\circ}{\sin 30^\circ}}$$

$$= 0.19$$

$$\phi_1 \approx 10^\circ$$

Velocity of Ball

$$v_B^F = \sqrt{v_0^2 - \frac{1}{5} v_p^{F\,2}} = 0.77 v_0$$

$$\vec{v}_b^F = 0.77 v_0 \left(\cos 10, \ \sin 10, \ 0\right)$$

$$\vec{v}_p^F = 1.44 v_0 \left(\cos 30, \ -\sin 30, \ 0\right)$$