Section Physical Pendulum

Simple Pendulum – Point mass, m, which rotates about a point O on a massless string of length ℓ.

$$I = m\ell^2$$

$$\omega = \sqrt{\frac{mg\ell}{I}} = \sqrt{g/\ell}$$

Physical Pendulum – A rigid body which rotates about a fixed pivot O, a distance ℓ from the CM.

Body Forces like gravity act ~~at CM~~ on all mass m_i in the body

$$\sum \vec{F}_i = M\vec{a}_{cm} = \sum m_i \vec{a}_i = \sum -m_i g\hat{j} = -Mg\hat{j}$$

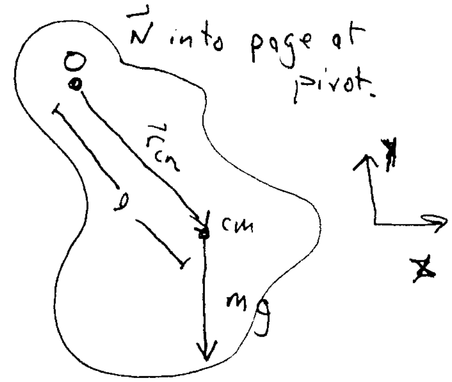
What about torques due to body forces?

$$\vec{N} = \sum \vec{r}_i \times m_i \vec{a}_i = \sum \vec{r}_i \times (-m_i g\hat{j})$$

$$= \left(\sum m_i \vec{r}_i\right) \times (-g\hat{j})$$

$$= M\vec{r}_{cm} \times (-g\hat{j}) = \vec{r}_{cm} \times (-Mg\hat{j})$$

So body forces exert a torque applied at CM.



EOM

$$\vec N = \frac{d\vec L}{dt} = I\omega\hat k = \vec r_{cm}\times(-Mg\hat y)$$

$$|\vec N| = |\vec r_{cm}\times Mg\hat k| = Mgl\sin\theta$$

(Minus because $-z$ into page)

$$\vec N = -Mgl\sin\theta\,\hat k$$

CEOM

$$I\dot\omega\hat k = -Mgl\sin\theta\,\hat k = I\ddot\theta\hat k \qquad (\omega = \dot\theta)$$

$$\boxed{\ddot\theta + \frac{Mgl}{I}\sin\theta = 0}$$

For small oscillation, Taylor expand $\sin\theta \sim \theta$

$$\ddot\theta + \frac{Mgl}{I}\theta = 0$$

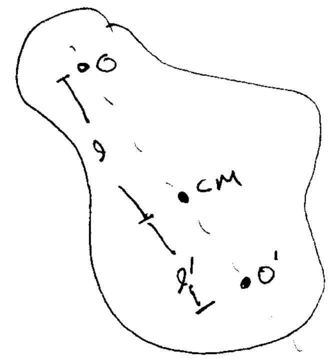
SHO $\quad \omega_0 = \sqrt{\frac{Mgl}{I}}$

As promised, for a simple pendulum $\quad I = ml^2$

$$\omega_0 = \sqrt{l/g}$$

Center of Oscillation (O') (with respect to O) — A

second point (in line with CM), where the oscillation will have same frequency as through O



By Parallel Axis Thm,

$$I_O = I_{cm} + m\ell^2$$

—or— Using radius of gyration

$$mk_O^2 = mk_{cm}^2 + m\ell^2$$

Frequency about O

$$\omega_O = \sqrt{\frac{mg\ell}{I}} = 2\pi f_O$$

$$T_O = 2\pi\sqrt{\frac{I}{mg\ell}} = 2\pi\sqrt{\frac{k_{cm}^2+\ell^2}{g\ell}}$$

If we select another pivot point O'

$$T_{O'} = 2\pi\sqrt{\frac{k_{cm}^2+\ell'^2}{g\ell'}}$$

For $T_O = T_{O'}$,

$$\frac{k_{cm}^2+\ell^2}{g\ell} = \frac{k_{cm}^2+\ell'^2}{g\ell'}$$

$$l'(K_{cm}^2 + l^2) = l(K_{cm}^2 + l'^2)$$

$$K_{cm}^2(l' - l) = l l'(l' - l)$$

$$K_{cm}^2 = l l'$$

$\Rightarrow$ l' locates center of oscillation.

* Does O' have to be in body?

Section - A completely general thm about torque

Recall that in general

$$\sum \vec{F}_i = m\dot{\vec{v}}_{cm}$$

$$\frac{d\vec{L}}{dt} = \vec{N} = \sum \vec{r}_i \times \vec{F}_i$$

$$\vec{L} = \vec{r}_{cm} \times m\vec{v}_{cm} + \sum \vec{r}_i{}' \times m_i \vec{v}_i{}'$$

$F_i \equiv$ external forces, $'$ about CM.

Now, write the torque in terms of torque about cm.

$$\vec{r}_i = \vec{r}_{cm} + \vec{r}_i{}' \qquad \vec{v}_i = \vec{v}_{cm} + \vec{v}_i{}'$$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\sum m_i(\vec{r}_{cm} + \vec{r}_i{}') \times (\vec{v}_{cm} + \vec{v}_i{}')$$

$$= \vec{N} = \sum(\vec{r}_{cm} + \vec{r}_i{}') \times \vec{F}_i$$

By defn of CM, $\sum m_i \vec{r}_i{}' = 0$, $\sum m_i \vec{v}_i{}' = 0$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}\left[\vec{r}_{cm} \times M\vec{v}_{cm} + \sum m_i \vec{r}_i{}' \times \vec{v}_i{}'\right]$$

$$= \vec{N} = \sum (\vec{r}_{cm} \times \vec{r}_i{}') \times \vec{F}_i$$

$$\overset{=0}{\vec{v}_{cm} \times M\vec{v}_{cm}} + \vec{r}_{cm} \times M\vec{a}_{cm} + \frac{d}{dt}\sum \vec{r}_i{}' \times m_i \vec{v}_i{}'$$

$$\cancel{+ \sum m_i \vec{v}_i{}' \times \vec{v}_i{}' + \sum m_i \vec{r}_i{}' \times \vec{a}_i}$$

$$= \vec{r}_{cm} \times \sum \vec{F}_i + \sum \vec{r}_i{}' \times \vec{F}_i$$

But $\sum \vec{F}_i = M\vec{a}_{cm}$

Which leaves.

$$\frac{d}{dt}\sum \vec{r}_i{}' \times m_i \vec{v}_i{}' = \sum \vec{r}_i{}' \times \vec{F}_i$$

Define Angular Momentum about CM

$$\bar{L} = \vec{L}' = \sum \vec{r}_i{}' \times m_i \vec{v}_i{}'$$

Torque About CM

$$\vec{N}' = \sum \vec{r}_i' \times \vec{F}_i$$

$$\frac{d\vec{L}'}{dt} = \vec{N}'$$

$\Rightarrow$ Completely general involves no additional assumptions beyond Newton.

$\Rightarrow$ Valid only for axis through CM.

Section - Laminar Motion of a Rigid Body

Laminar Motion - All particles move in a plane parallel to some axis.

In General

$$\vec{F}_{ext} = M \vec{a}_{cm}$$

$$\frac{d\vec{L}'}{dt} = \vec{N}'$$

Since axis of rotation is fixed,

$$|\vec{N}'| = \left| \frac{d\vec{L}'}{dt} \right| = I_{cm} \dot{\omega}$$

Recall Friction

Normal Force.

Static Friction - $|F_s| \leq \mu_s F_N$

$\mu_s \equiv$ Coefficient of Static Friction

$= \tan\theta$, the angle slipping begins.

$\Rightarrow$ Point of application of force does not ~~change~~, move, so no work done.

$\Rightarrow$ Do not know magnitude of force.

<u>Dynamic (kinetic) Friction</u> $|F_K| = \mu_K F_N$

$\Rightarrow$ We know force.

$\Rightarrow$ Point of application of force moves, force does work, cannot use conservation of energy.

<u>Condition of Rolling</u> $\vec{v}_{cm}$ = velocity of contact point.

$$v_{cm} = a\,\omega$$

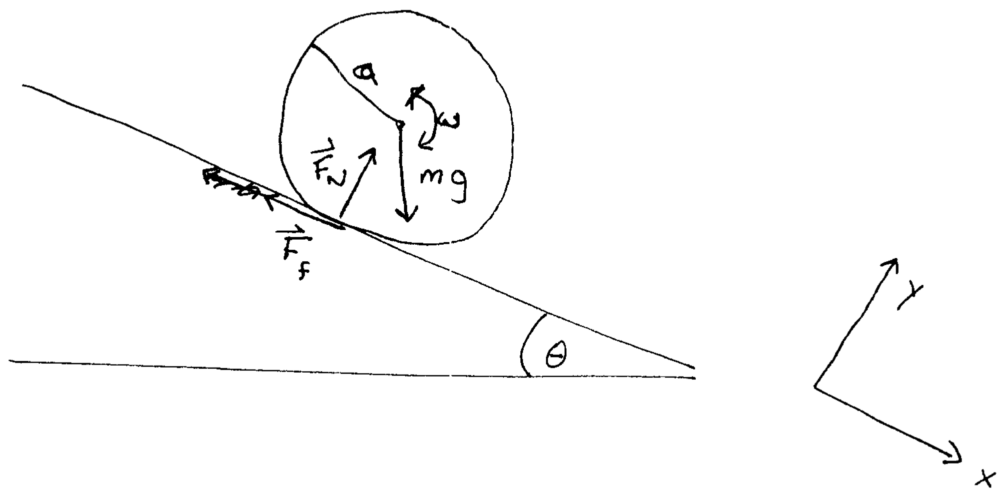
radius

<u>Relation of μ_s and μ_K</u> — $\mu_K < \mu_s$

$\Rightarrow$ Tip on incline until sliding starts, if $\mu_K > \mu_s$ sliding would immediately stop.

$\Rightarrow$ Once you get something moving it is easier to keep it moving.

Example Object rolling down inclined plane,



EOM

$$\vec{F}_{ext} = M\vec{a}_{cm} = \vec{F}_g + \vec{F}_N + \vec{F}_f$$

$$\vec{N}' = \frac{d\vec{L}'}{dt} = I_{cm}\dot{\omega} = |F_f|a$$

CEOM

$$M\ddot{x}_{cm} = Mg\sin\theta - F_f$$

$$M\ddot{y}_{cm} = -Mg\cos\theta + F_N = 0$$

$$\Rightarrow F_N = Mg\cos\theta$$

To determine $|F_f|$ we need to know if slipping occurs, that is does $|F_f| > \mu_s F_N$

Since $\mu_k < \mu_s$, we can answer the question does the object slip by working the problem with the object not slipping, then testing the frictional force against $\mu_s F_N$.

<u>No Slipping</u>

$$v_{cm} = a\omega$$

$$a_{cm} = a\dot{\omega}$$

$$F_f = \frac{I_{cm} a_{cm}}{a^2} \qquad \text{(Torque EQN)}$$

<u>Substitute into x - CEOM</u> -

$$M a_{cm} = Mg \sin\theta - \frac{I_{cm} a_{cm}}{a^2}$$

$$a_{cm}\left(M + \frac{I_{cm}}{a^2}\right) = Mg \sin\theta$$

$$a_{cm} = \frac{Mg \sin\theta}{M + \frac{I_{cm}}{a^2}}$$

Does slipping occur?

$$F_f < \mu_s F_N = \mu_s Mg\cos\theta$$

$$F_f = \frac{I\,a_{cm}}{a^2} = \frac{Mg\sin\theta}{1 + \frac{Ma^2}{I}} < \mu_k Mg\cos\theta$$

Slipping begins at θ_s where $F_f = \mu_k F_N$

$$\mu_k = \frac{\tan\theta_s}{1 + a^2/k_{cm}^2}$$

When does slipping stop?

The point of contact stops slipping if $v_{cm} = a\omega$

$\underline{\theta < \theta_s}$ No slipping

x- CEOM

$$M a_{cm} = Mg \sin\theta - \frac{I_{cm} a_{cm}}{a^2}$$

$$a_{cm} = \frac{Mg \sin\theta}{M + \frac{I_{cm}}{a^2}} = \text{constant}$$

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_{cm} t^2$$

$$y(t) = 0$$

$\underline{\theta > \theta_s}$

$$F_f = \mu_K Mg \cos\theta$$

$$M a_{cm} = Mg \sin\theta - F_f$$

$$= Mg \sin\theta - \mu_K Mg \cos\theta$$

$$a_{cm} = g(\sin\theta - \mu_K \cos\theta) = \text{constant}$$

But we're not done

$$a F_f = I_{cm} \dot{\omega}$$

$$\dot{\omega} = \frac{a}{I_{cm}} \left(Mg \mu_k \cos\theta \right) = \text{constant}$$

$$\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2} \dot{\omega} t^2$$

Energy considerations

If the ball is not slipping, then energy is conserved.

$$E = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I \omega^2 + mgh$$

$$h(x) = -x \sin\theta$$

$$\omega = \frac{v_{cm}}{a}$$

Lecture 4/5

EOM $$\sum_i \vec{F}_i = M\vec{a}_{cm}$$

$$\vec{N}' = \frac{d\vec{L}'}{dt} = I\dot{\omega}$$

Static Friction $F_s \leq \mu_s F_N$

$\Rightarrow$ No work

$\Rightarrow \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\omega^2 + V = \text{constant} \Rightarrow a\omega = v$

Dynamic Friction $F_k = \mu_k F_N$

$\Rightarrow$ Energy Lost to environment

Condition of Rolling $\vec{v}_{cm} = a\omega$

$\Rightarrow$ Relation μ_k, μ_s $\mu_k < \mu_s$

$\Rightarrow$ An object starts slipping when $F_f > \mu_s F_N$

$\Rightarrow$ An object stops slipping when the condition of rolling is met again.

Section – Impulse and Rotation

Linear Impulse $\Delta\vec{p} = m\,\Delta\vec{v}_{cm} = \int \vec{F}\,dt$

Rotational Impulse $I_{cm}\,\Delta\omega = \int |\vec{N}|\,dt = \Delta L'$ ~~cm~~

$\Rightarrow$ For free rotation, the axis must be cm.

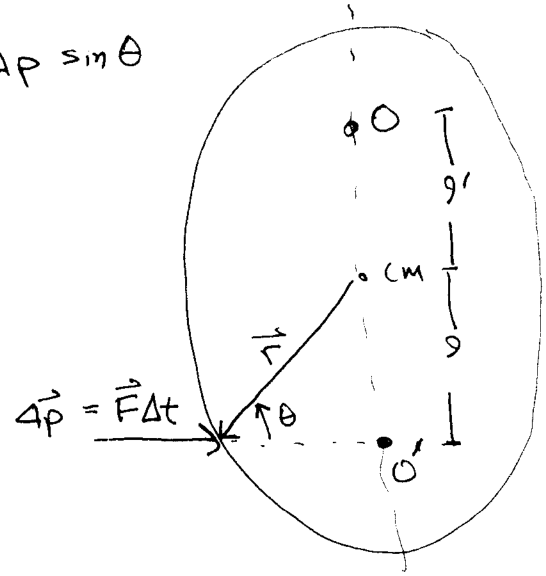
$\Rightarrow$ For constrained rotation about a point, the axis (~~the~~ and moment of inertia) about point may be used.

$\Rightarrow$ If an object is struck so that the linear impulse is $\perp$ to a line from the cm (or the fixed axis of rotation) the angular impulse $|\vec{r} \times \Delta\vec{p}| = \Delta p\,l$ where l is the distance (along the axis through cm)

$|\Delta \vec{L}| = |\vec{r} \times \Delta \vec{p}| = r \Delta p \sin\theta$

but $\ell = r\sin\theta$

$|\Delta L| = (\Delta p)\ell$



<u>Center of Percussion</u> – If an object is struck, ~~at a point~~ perpendicular to a line through a center of mass. The center of percussion is the point that is left at rest after the object is struck. $\Rightarrow$ If you were holding the object at the center of percussion, you feel no impulse.

If the object is initially at rest, the velocity of an ~~object~~ point immediately after impact is

$$V_O = V_{cm} - \ell'\omega$$

Linear Impulse

$$\Delta p = M \Delta v_{cm} = M v_{cm} \Rightarrow v_{cm} = \frac{\Delta p}{M}$$

Angular Impulse

$$\Delta L = \Delta p\, l = I_{cm} \Delta \omega = I_{cm}\, \omega$$

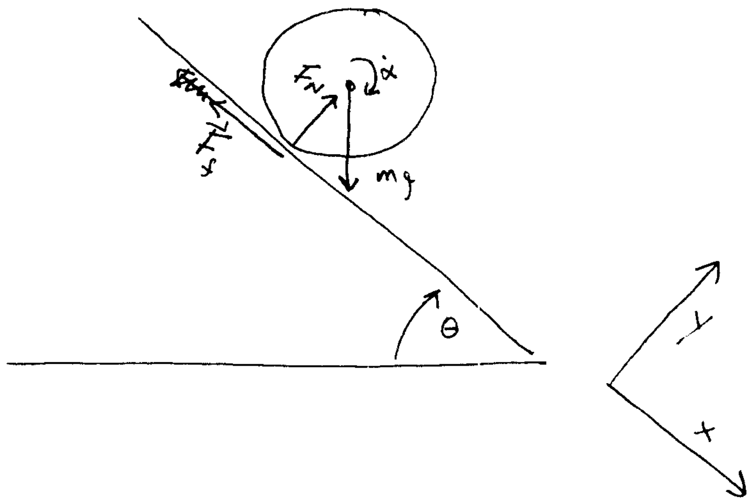
$$\omega = \frac{\Delta p\, l}{I_{cm}}$$

$$v_0 = \frac{\Delta p}{M} - \frac{\Delta p\, l\, l'}{I_{cm}} = 0 \quad \text{for center of percussion}$$

$$\frac{I_{cm}}{M} = l\, l' = k_{cm}^2$$

$\Rightarrow$ Center of Percussion is the same point as center of oscillation

Example   Ball Rolling Down Plane



EOM

$$\sum_i \vec{F}_i = M\ddot{\vec{x}}_{cm} = \vec{F}_g + \vec{F}_N + \vec{F}_f$$

$$\vec{N}' = \frac{d\vec{L}'}{dt} = \vec{a} \times \vec{F}_f = I_{cm}\dot{\omega}\hat{k}$$

CEOM   x-component

$$M\ddot{x} = Mg\sin\theta - F_f$$

y-component

$$M\ddot{y} = -Mg\cos\theta + F_N = 0$$

$$\Rightarrow \quad F_N = Mg\cos\theta$$

Rotation

$$aF_f = I\dot{\omega}$$

Do I know F_f? $\Rightarrow$ No F_f depends on whether the object is slipping or not slipping.

Where can I go from here?

I. Give object slipping.

II. Give object rolling.

III. Ask is object slipping

I. IF Slipping $\quad F_f = \mu_K F_N = \mu_K Mg\cos\theta$

$$M\ddot{x}_{cm} = Mg\sin\theta - \mu_K Mg\cos\theta$$

$$\ddot{x}_{cm} = g(\sin\theta - \mu_K\cos\theta) = \text{constant}$$

$$x_{cm}(t) = x_0 + v_0 t + \frac{1}{2}(\ddot{x}_{cm})t^2$$

$\Rightarrow$ Tells us nothing about rotation.

$$a F_f = I_{cm}\dot{\omega}$$

$$\dot{\omega} = \frac{a\mu_K Mg\cos\theta}{I_{cm}} = \text{constant}$$

$$\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\dot{\omega}t^2$$

II. No Slipping $\Rightarrow$ F_f unknown, will find.

Condition of Rolling $\cancel{}\ \dot{x} = a\omega$

$$F_f = \frac{I_{cm}\dot{\omega}}{a} = \frac{I_{cm}\ddot{x}}{a^2}$$

$$M\ddot{x} = Mg\sin\theta - \frac{I_{cm}\ddot{x}}{a^2}$$

$$\left(M + \frac{I_{cm}}{a^2}\right)\ddot{x} = Mg\sin\theta$$

$$\ddot{x} = \frac{Mg\sin\theta}{M + \frac{I_{cm}}{a^2}} = \frac{g\sin\theta}{1 + \frac{I_{cm}}{Ma^2}} = \text{constant}$$

$$x(t) = x_0 + v_0 t + \frac{1}{2}(\ddot{x})t^2$$

$$\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}(\dot{\omega})t^2$$

$$\dot{\omega} = \frac{\ddot{x}}{a} = \text{constant}$$

III. Does it slip?

$$F_f = \frac{I_{cm}\ddot{x}}{a^2} = \frac{Mg\sin\theta}{1 + \frac{Ma^2}{I_{cm}}} = \mu_s F_N$$

$$\frac{Mg \sin\theta}{1 + \frac{Ma^2}{I_{cm}}} = \mu_s Mg \cos\theta$$

$$\mu_s = \frac{\tan\theta}{1 + \frac{Ma^2}{I_{cm}}}$$