The Big Picture (Chap 7+8)

Haley asked for a talk about the big picture in Chapters 7 and 8.

Both chapters deal with the effects of having more than one mass moving in the system. In chapter 7 the masses can have any relationship. In chapter 8, we studied the case where the distances between the masses were unchanging.

For any system of masses,

$$\sum F_i = \dot{\vec{P}} = m\vec{a}_{cm}$$

$$\frac{d\vec{L}}{dt} = \vec{N} = \sum \vec{r}_i \times \vec{F}_i$$

$$\vec{L} = M\vec{r}_{cm} \times \vec{V}_{cm} + \sum \vec{r}_i{}' \times m_i \vec{v}_i{}'$$

$$T = \frac{1}{2} m v_{cm}^2 + \sum_i \frac{1}{2} m_i v_i^{2\prime}$$

Two Mass — EOM can be reduced to

$$\mu \frac{d^2 \vec{r}_{12}}{dt^2} = f(r_{12})\, \hat{r}_{12}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Collisions (2 particle) — Momentum Conserved, Energy Loss Q

(1) Head On

(2) Lab System

(3) Center-of-mass system.

Variable Mass ~~Sys~~ The masses which belong to to system are changing

N particle systems where the internal forces are unknown or non-existent.

Chap. <u>Rigid Bodys</u> (Special Case of Chap 7)

$$\sum F_i = M\vec{a}_{cm}$$

$$\frac{d\vec{L}'}{dt} = \vec{N}'$$

Because of the addition constraint of a rigid body, the dynamical effect of the extended mass can be captured in a single number.

If body is a lamina, flat

<u>Moment of Inertia</u> - $I_z = \sum m_i (x_i^2 + y_i^2) = Mk^2$

$$I_o = I_{cm} + ml^2$$

<u>EOM</u>

$$\vec{L}' = I\vec{\omega}$$

$$\frac{d\vec{L}'}{dt} = I\dot{\vec{\omega}} = \vec{N}'$$

$$T_{rot} = \frac{1}{2} I \omega^2$$

<u>Impulse</u>

$$\Delta\vec{p} = \int \vec{F} dt = M\Delta\vec{v}_{cm}$$

$$\Delta\vec{L} = \int \vec{N} dt = I\Delta\omega$$

From this we examine,

~~Rod~~ Center of Osilation

Center of Percussion

Condition of Rolling $v_{cm} = a\omega$

Pendula

Lecture 4/9

- Test Thursday 17 6:00 pm somewhere.
  Chapters 7-9.

- Work problem 9.3 by diagonalizing matrix.

- I did not recieve homework 7 from 2 stewarts, chris.

- Dr. Harters toys.

Section - The Big Picture Chap 9

If the body is not laminar, but is rigid.

$$\overleftrightarrow{I} = \begin{pmatrix} I_{xx} & & \ddots \\ I_{yx} & I_{yy} & \\ \vdots & & I_{zz} \end{pmatrix}$$

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} \quad \& \qquad \frac{d\vec{L}}{dt} = \vec{N}$$

$$T_{rot} = \frac{1}{2}\, \vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega}$$

$\Rightarrow$ Principle Moments - In some coordinate system $\overleftrightarrow{I}$ diagonal.

$\Rightarrow$ Euler's Eqns - Write EOM in a frame rotating with the body in the coordinate system of the principle moments.

$\Rightarrow$ Euler Angles - Relate rotating coordinate system to fixed system.

Special Cases

I. Free Rotation.

II. Gyroscope

III. Stable Rolling.

Section – Inertia

Moments of Inertia (About origin)

$$I_{xx} = \sum m_i (y_i^2 + z_i^2)$$

$$I_{yy} = \sum m_i (x_i^2 + z_i^2)$$

$$I_{zz} = \sum m_i (x_i^2 + y_i^2)$$

Products of Inertia

$$I_{xy} = I_{yx} = -\sum m_i x_i y_i$$

$$I_{yz} = -I_{zy} = -\sum m_i y_i z_i$$

$$I_{xz} = I_{zx} = -\sum m_i x_i z_i$$

Inertia Tensor (Matrix with addition transformation properties).

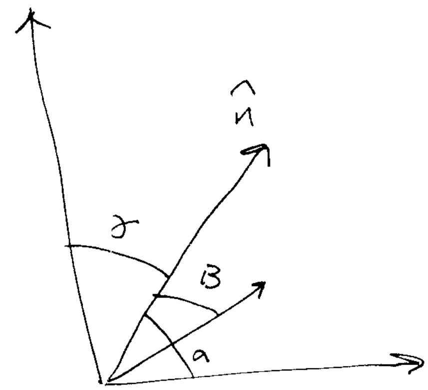
$$\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

Symmetric

Moment of Inertia about axis $\hat{n}$

$$\hat{n} = (\cos\alpha, \cos\beta, \cos\gamma)$$

Direction cosines.



$$\boxed{I = \hat{n}^T \, I \, \hat{n}} = \hat{n}^T \cdot \overleftrightarrow{I} \cdot \hat{n}$$

number

$$I_{\hat{n}} = \begin{pmatrix} \cos\alpha & \cos\beta & \cos\gamma \end{pmatrix} \begin{pmatrix} & & \\ & I & \\ & & \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix}$$

n^T is the "transpose" (flipped over through the diagonal) of n.

<u>Displaced Axis Thm</u> — The moment of inertia about an origin displaced by a vector $\vec{a}$ from the center of mass is

$$\overleftrightarrow{I}_{\vec{a}} = \overleftrightarrow{I}_{cm} + M\left(a^2 \delta_{\alpha\beta} - a_\alpha a_\beta\right)$$

$$= \overleftrightarrow{I}_{cm} + M \begin{pmatrix} a_y^2 + a_z^2 & -a_x a_y & -a_x a_z \\ & a_x^2 + a_z^2 & a^2 - a_y a_z \\ & & a_x^2 + a_y^2 \end{pmatrix}$$

$\delta_{\alpha\beta} = 1$ if $\alpha = \beta$, 0 otherwise.

$\vec{a} = (a_x, a_y, a_z)$ $\qquad a^2 = a_x^2 + a_y^2 + a_z^2$

$$\overleftrightarrow{I}_{\vec{a}} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} + M \begin{pmatrix} (a_y^2 + a_z^2), & -a_x a_y, & -a_x a_z \\ -a_y a_x & a_x^2 + a_z^2 & -a_y a_z \\ & & a_x^2 + a_y^2 \end{pmatrix}$$

Section – Using Moment of Inertia

$$\vec{N} = \frac{d\vec{L}}{dt} \qquad \text{Always}$$

$$\boxed{\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}} \qquad \text{Angular Momentum}$$

$\Rightarrow$ Problem: As the body rotates the moment of inertia changes with time.

$\Rightarrow$ $\vec{L}$ is not parallel to $\vec{\omega}$ in general.

<u>Rotational Energy</u> $\quad T_{rot} = \frac{1}{2}\vec{\omega}\cdot\vec{L}$

In analogy to $T_{trans} = \frac{1}{2}pv$

$$T_{rot} = \frac{1}{2}\omega^T I \omega$$

$$= \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega}$$

Section – Principle Axes

<u>Principle Moments</u> – A coordinate system always exists such that

$$\underline{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

I_i are the principle moments.

In this frame,

$$\underline{I} = n^T \overleftrightarrow{I} n = I_1 \cos^2\alpha + I_2 \cos^2\beta + I_3 \cos^2\gamma$$

$$\vec{L} = \overleftrightarrow{I}\omega = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}$$

$$T_{rot} = \frac{1}{2}\,\omega^T \underline{I}\,\omega = \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

<u>Static Balancing</u> – CM on axis of rotation.

<u>Dynamic Balancing</u> – $\vec{\omega}$ along principle axis, so $\vec{L}$ points in $\vec{\omega}$ direction.

Diagonalizing a Matrix - Looking for a transformation matrix A so that the inertia tensor in the ' coordinate system is diagonal.

$$I' = A^{-1} I A \quad \text{diagonal}$$

If x_1' is the direction of one of the principle moments, the $\vec{x}_i'$ are the basis vectors for the diagonal system $(x_i' = \cdot\cdot 1 \cdot\cdot)$

$$I' \vec{x}_1' = \lambda_1 \vec{x}_1'$$

Matrix equations don't change under transformation.

$$I x_1 = \lambda_1 x_1$$

$$(I - \lambda) x_1 = 0$$

A solution exists if $\det(I - \lambda) = 0$, λ_i are the eigenvalues of the inertia tensor and also the principle moments of inertia.

The eigen vectors are the direction of the principle axes.

The transformation matrix A that converts to the diagonal coordinate system is:

eigenvectors

$$A = \begin{pmatrix} x_{1x} & x_{2x} & x_{3x} \\ x_{1y} & x_{2y} & x_{3y} \\ x_{1z} & x_{2z} & x_{3z} \end{pmatrix}$$

Example - 2x2 Eigenvectors

(Inertia tensor symmetry)

$$I = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

$$I \vec{x} = \lambda \vec{x}$$

$$I - \lambda = \begin{pmatrix} 1-\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix}$$

$$\text{Det}(I-\lambda) = (1-\lambda)(3-\lambda) - 4 = 0$$

$$3 - 4\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 4\lambda - 1 = 0$$

$$\frac{4 \pm \sqrt{16+4}}{2} \qquad b^2 - 4ac$$

$$\lambda_{21} \quad 2 \pm \sqrt{5}$$

$$\lambda_1 = 2 + \sqrt{5}$$

$$\lambda_2 = 2 - \sqrt{5}$$

$\lambda_1 = 2 + \sqrt{5}$

$$\begin{pmatrix} 1 - (2+\sqrt{5}) & 2 \\ 2 & 3 - (2+\sqrt{5}) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$(-1 - \sqrt{5})\,x + 2y = 0$$

$$2x + (1 - \sqrt{5})\,y = 0$$

Let $y = 1$ (Since equation linear, can choose one coordinate)

$$x = -\tfrac{1}{2}(1 - \sqrt{5}) = +0.618$$

$$|\vec{x}_1| = 1.176$$

$$\hat{x}_1 = (+0.526,\ 0.851)$$

$\lambda_2 = 2 - \sqrt{5}$

$$\begin{pmatrix} 1-(2-\sqrt{5}) & 2 \\ 2 & 3-(2-\sqrt{5}) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$(-2+\sqrt{5})x + 2y = 0$$

$$2x + (1+\sqrt{5})y = 0$$

$\cancel{x=1}$ $\quad$ $\cancel{y = -\frac{1}{2}}$ $\quad$ $y = 1$ $\quad$ $x = -\frac{1}{2}(1+\sqrt{5})$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = -1.62$

$|x_2| = 1.90$

$\hat{x}_2 = (-0.853, 0.526)$

$\hat{x}_1 \cdot \hat{x}_2 = 0$ ✓

Section - Euler's Equations

Everything is simplier in a frame rotating with the body. Try writing the EOM in that frame.

$$\vec{N} = \frac{d\vec{L}}{dt} \qquad \text{always.}$$

$$\vec{N} = \left(\frac{d\vec{L}}{dt}\right)_{rot} + \vec{\omega} \times \vec{L}$$

But $\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$

$$= \left(\frac{dI}{dt}\right)_{rot} \cdot \vec{\omega} + \overleftrightarrow{I} \cdot \left(\frac{d\vec{\omega}}{dt}\right)_{rot} + \vec{\omega} \times \left(\overleftrightarrow{I} \cdot \vec{\omega}\right)$$

$$\left(\frac{d\omega}{dt}\right)_{rot} = \left(\frac{d\omega}{dt}\right)_{fixed} \equiv \mathring{\vec{\omega}} \qquad \text{because } \vec{\omega} \times \vec{\omega} = 0$$

$$\left(\frac{d\overleftrightarrow{I}}{dT}\right)_{rot} = 0$$

because we defined the frame ~~was defined~~ to rotate with the body.

$$\vec{N} = \overleftrightarrow{I}^{P} \cdot \mathring{\vec{\omega}} + \vec{\omega} \times \left(\overleftrightarrow{I}^{P} \cdot \vec{\omega}\right)$$

$$\overleftrightarrow{I}^{P} \cdot \vec{\omega} = \begin{pmatrix} I_1 \dot{\omega}_1 \\ I_2 \dot{\omega}_2 \\ I_3 \dot{\omega}_3 \end{pmatrix}$$

$$\vec{\omega} \times \left(\overleftrightarrow{I} \cdot \vec{\omega} \right) = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ I_1 \omega_1 & I_2 \omega_2 & I_3 \omega_3 \end{vmatrix}$$

$$= \begin{pmatrix} \omega_2 \omega_3 (I_3 - I_2) \\ \omega_1 \omega_3 (I_1 - I_3) \\ \omega_1 \omega_2 (I_2 - I_1) \end{pmatrix}$$

<u>Euler's Eqn</u>

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} I_1 \dot{\omega}_1 \\ I_2 \dot{\omega}_2 \\ I_3 \dot{\omega}_3 \end{pmatrix} + \begin{pmatrix} \omega_2 \omega_3 (I_3 - I_2) \\ \omega_1 \omega_3 (I_1 - I_3) \\ \omega_1 \omega_2 (I_2 - I_1) \end{pmatrix}$$

* All quantities in referarence to principle axis frame.

What can we do with these? I $\vec{\omega}$ known

we can find torques (Prb. 9.6).

Suppose no torque $\Rightarrow$ Free Rotation $N_i = 0$

Still bad.

Suppose the body has an axis of symmetry, so that $I_1 = I_2 = I$ and $I_3 = I_s$

$$0 = I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_s - I)$$

$$0 = I_2 \dot{\omega}_2 + \omega_1 \omega_3 (I - I_s)$$

$$0 = I_3 \dot{\omega}_3$$

$$\omega_3 = \text{constant}$$

$$\cancel{\dot{\omega}_1 = \omega_2}$$

$$\dot{\omega}_1 + \omega_2 \left[\frac{\omega_3 (I_s - I)}{I} \right] = 0$$

$$\dot{\omega}_2 - \omega_1 \left[\frac{\omega_3 (I_s - I)}{I} \right] = 0 \qquad *$$

We know how to solve this,

$$\ddot{\omega}_1 + \dot{\omega}_2 \left[\Omega \right] = 0 \qquad \Omega = \frac{\omega_3 (I_s - I_1)}{I}$$

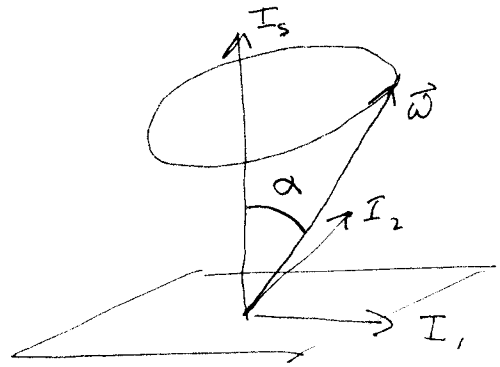
Substitute $*$

$$\ddot{\omega}_1 + \Omega^2 \omega_1 = 0$$

$$\omega_1 = A \cos \Omega t$$

$$\omega_2 = A \sin \Omega t$$

$\Longrightarrow$ The 1,2 component of the instantaneous axis of rotation describe a circle about the symmetry axis $\Rightarrow$ Body Cone.



$$\omega_3 = \omega \cos\alpha$$

$$\Omega = \left(\frac{I_s}{I} - 1\right) \cos\alpha$$

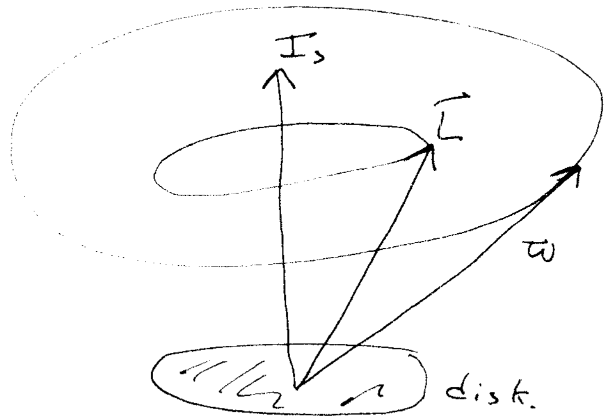
All these quantities were body fixed. We have one quantity which is space fixed, the angular momentum, $\vec{L}$.

The direction of the angular momentum is called the invariable line (Prb. 9.9). The symmetry axis also describes a cone about the angular momentum. The rate of this precession is

$$\dot{\phi} = \omega \left[1 + \left(\frac{I_s^2}{I^2} - 1\right) \cos^2\alpha\right]^{1/2}$$

where α is angle between $\vec{\omega}$ and I_s.

This is the wobble of an object.



I_3
$\vec{L}$
ω
disk.

Lecture 4/13

- Test Thursday April 17 Kimpel 105
  6:00pm.

- Formula sheet only.

Section - Free Rotations

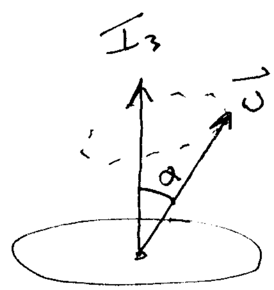
Free Rotation $\Rightarrow$ Torque Free.

Symmetric Top $I_3 = I_s$ $I_1 = I_2 = I$

$\Rightarrow$ Two precession rates -

(1) $\vec{\omega}$ about I_s $\Rightarrow$ ~~space~~ body cone (Ω)

(2) $\vec{L}$ about I_s $\Rightarrow$ Space cone. ($\dot{\phi}$)



$\Rightarrow$ Angle α is angle between I_s and $\vec{\omega}$.

Precession $\vec{\omega}$ about I_s

$$\Omega = \omega\left(\frac{I_s}{I} - 1\right)\cos\alpha$$

Precession $\vec{L}$ about I_s (wobble)

$$\dot{\phi} = \omega\left[1 + \left(\frac{I_s^2}{I^2} - 1\right)\cos^2\alpha\right]^{1/2}$$

Section – Lagranges Equations

$\boxed{\text{Different formulation of Newton's Laws.}}$

Generalized Coordinates – Any set of coordinates q_i that specify the location of the masses, m_i in the system.

Lagrangian (L) $\quad L = T - V$

Constraints (Holonomic) $\quad f(x, y, z, t) = 0$

* No velocities.

Degrees of Freedom – N masses,

$$3N - m$$

where m is the number of constraints.

$\Longrightarrow$ One q for each degree of freedom.

Lagrange's Eqns

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

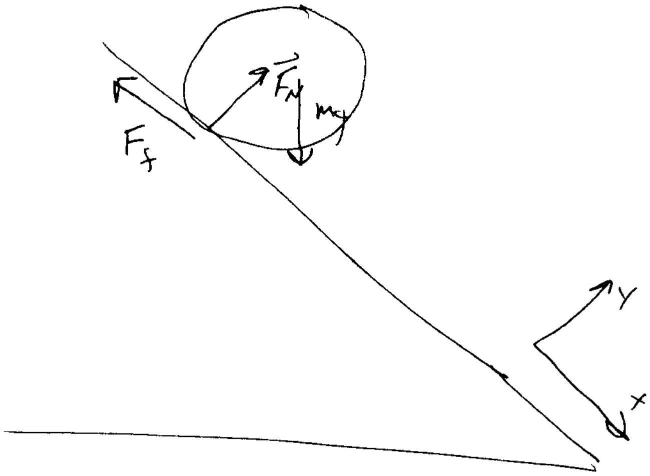
Lecture 4/18

- Clarifications on take home problem
- Talk Yulia about grading
- Evaluations.

$$\boxed{\text{Final Exam} - \text{Tuesday May 6 from 3-5 here}}$$

- Hint problem 4 $\Rightarrow$ if I ask if its elastic in the last part you may not assume its elastic in the first part.

Section – Exam



An object slips if $F_f \geq \mu_s F_N$

$\Longrightarrow$ Try rolling without slipping $\Rightarrow$ $\omega = v/a$

compute F_f and see if it slips.

x – comp

$$m\ddot{x} = mg\sin\theta - F_f$$

y - comp

$$0 = mg\cos\theta + F_N$$

Torque

$$F_f\, a = \frac{dL}{dt} = I\dot{\omega} = \frac{I}{a}\ddot{x} \quad (\text{if rolling})$$

$$F_f = \frac{I}{a^2}\ddot{x}$$

$$\left(m + \frac{I}{a^2}\right)\ddot{x} = mg\sin\theta$$

$$\ddot{x} = \frac{mg \sin\theta}{\left(m + \frac{mk^2}{d^2}\right)} = \frac{g \sin\theta}{\left(1 + \frac{k^2}{d^2}\right)}$$

If $\quad F_f = \dfrac{\frac{I}{d^2}\, g \sin\theta}{1 + k^2/d^2} < \mu_s mg \cos\theta$

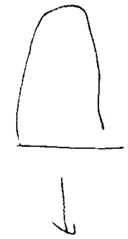
<u>slips</u>

$$= \frac{\sin\theta}{2} < \mu_s \cos\theta$$

$$\tan\theta < 2 \tan 60^\circ$$

$$\theta = 73^\circ$$

$$\vec{F}_{ext} + \dot{m} V_{rel} = m\vec{a}_{cm}'$$



$$-mg + \dot{m} \overset{2PA}{V_{rel}} = m \frac{dv'}{dt}$$

$$m(t) = M_w + M_r - \gamma t$$

$$\frac{dv'}{dt} = -g + \frac{2PA}{M_w + M_r - \gamma t}$$

$$v(t) = -gt + \int_0^t \frac{2PA}{M_w + M_r - \gamma t} \, d\tau$$

$$\int \frac{du}{u} = \ln u/u_0$$