Homework 1 – Mechanics -Due Tuesday 1/21/2003

Turn Homework in at Dr. Stewart's Mailbox in the Physics Office

Fowles Problems

1.4
1.14
1.16
1.18
1.19
1.20
1.22
1.23

Review Problems

R1. Draw the free body diagram and the extended free body diagram for a marble sliding into a bowl. Does the marble rotate as it slides? Justify your answer.

R2. A hockey puck (Puck 1) of mass m and velocity v collides elastically with a second hockey puck (Puck 2) on a frictionless surface. The second puck is attached to a stake by a massless string of length R. The direction of motion of puck 1 and the string are at right angles. After the collision, the hockey puck revolves around the stake. The three objects of interest in this problem are the two hockey pucks and the earth. The earth is assumed to be at rest before the collision.

1. **Analyze the energy, momentum, and angular momentum using the universe as the system.**
2. **Analyze the energy, momentum, and angular momentum using the second hockey puck as the system.**

Bonus

B1. What are the classic blunders?

− 1/4 for terrible work

gives directly

$$\mathbf{a} = (-b\omega_2^2 \sin^2\theta - b\omega_1^2)\mathbf{e}_r - b\omega_2^2 \sin\theta \cos\theta \,\mathbf{e}_\theta + 2b\omega_1\omega_2 \cos\theta \,\mathbf{e}_\phi$$

The point at the top has coordinate $\theta = 0$, so at that point

$$\mathbf{a} = -b\omega_1^2\mathbf{e}_r + 2b\omega_1\omega_2\mathbf{e}_\phi$$

The first term on the right is the centripetal acceleration, and the last term is a transverse acceleration normal to the plane of the wheel.

PROBLEMS

1.1 Given the two vectors $\mathbf{A} = \mathbf{i} + \mathbf{j}$ and $\mathbf{B} = \mathbf{j} + \mathbf{k}$, find the following:

(a) $\mathbf{A} + \mathbf{B}$ and $|\mathbf{A} + \mathbf{B}|$

(b) $3\mathbf{A} - 2\mathbf{B}$

(c) $\mathbf{A} \cdot \mathbf{B}$

(d) $\mathbf{A} \times \mathbf{B}$ and $|\mathbf{A} \times \mathbf{B}|$

1.2 Given the three vectors $\mathbf{A} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{B} = \mathbf{i} + \mathbf{k}$, and $\mathbf{C} = 4\mathbf{j}$, find the following:

(a) $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C})$ and $(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C}$

(b) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$

(c) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ and $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$

1.3 Find the angle between the vectors $\mathbf{A} = a\mathbf{i} + 2a\mathbf{j}$ and $\mathbf{B} = a\mathbf{i} + 2a\mathbf{j} + 3a\mathbf{k}$. (*Note:* These two vectors define a face diagonal and a body diagonal of a rectangular block of sides a, $2a$, and $3a$.)

1.4 Consider a cube whose edges are each of unit length. One corner coincides with the origin of an $Oxyz$ Cartesian coordinate system. Three of the cube's edges extend from the origin along the positive direction of each coordinate axis. Find the vector that begins at the origin and extends

(a) along a major diagonal of the cube;

(b) along the diagonal of the lower face of the cube.

(c) Calling these vectors $\mathbf{A}$ and $\mathbf{B}$, find $\mathbf{C} = \mathbf{A} \times \mathbf{B}$.

(d) Find the angle between $\mathbf{A}$ and $\mathbf{B}$.

1.5 Assume that two vectors $\mathbf{A}$ and $\mathbf{B}$ are known. Let $\mathbf{C}$ be an unknown vector such that $\mathbf{A} \cdot \mathbf{C} = u$ is a known quantity and $\mathbf{A} \times \mathbf{C} = \mathbf{B}$. Find $\mathbf{C}$ in terms of $\mathbf{A}$, $\mathbf{B}$, u, and the magnitude of $\mathbf{A}$.

1.6 Given the time-varying vector

$$\mathbf{A} = \mathbf{i}\alpha t + \mathbf{j}\beta t^2 + \mathbf{k}\gamma t^3$$

where α, β, and γ are constants, find the first and second time derivatives $d\mathbf{A}/dt$ and $d^2\mathbf{A}/dt^2$.

1.7 For what value (or values) of q is the vector $\mathbf{A} = \mathbf{i}q + 3\mathbf{j} + \mathbf{k}$ perpendicular to the vector $\mathbf{B} = \mathbf{i}q - q\mathbf{j} + 2\mathbf{k}$?

1.8 Give an algebraic proof and a geometric proof of the following relations.

$$|\mathbf{A} + \mathbf{B}| \le |\mathbf{A}| + |\mathbf{B}|$$
$$|\mathbf{A} \cdot \mathbf{B}| \le |\mathbf{A}|\,|\mathbf{B}|$$

1.9 Prove the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$.

1.10 Two vectors **A** and **B** represent concurrent sides of a parallelogram. Show that the area of the parallelogram is equal to $|\mathbf{A} \times \mathbf{B}|$.

1.11 Show that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is not equal to $\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C})$.

1.12 Three vectors **A, B,** and **C** represent three concurrent edges of a parallelepiped. Show that the volume of the parallelepiped is equal to $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$.

1.13 Verify the transformation matrix for a rotation about the z-axis through an angle ϕ followed by a rotation about the y'-axis through an angle θ, as given in Example 1.8.2.

1.14 Express the vector $2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ in the primed triad $\mathbf{i}'\mathbf{j}'\mathbf{k}'$ in which the $x'y'$-axes are rotated about the z-axis (which coincides with the z'-axis) through an angle of 30°.

1.15 Consider two Cartesian coordinate systems $Oxyz$ and $Ox'y'z'$ that initially coincide. The $Ox'y'z'$ undergoes three successive counterclockwise 45° rotations about the following axes: first, about the fixed z-axis; second, about its own x'-axis (which has now been rotated); finally, about its own z'-axis (which has also been rotated). Find the components of a unit vector **X** in the $Oxyz$ coordinate system that points along the direction of the x'-axis in the rotated $Ox'y'z'$ system. (*Hint: It would be useful to find three transformation matrices that depict each of the above rotations. The resulting transformation matrix is simply their product.*)

1.16 A racing car moves on a circle of constant radius b. If the speed of the car varies with time t according to the equation $v = ct$, where c is a positive constant, show that the angle between the velocity vector and the acceleration vector is 45° at time $t = \sqrt{b/c}$. (*Hint: At this time the tangential and normal components of the acceleration are equal in magnitude.*)

1.17 A small ball is fastened to a long rubber band and twirled around in such a way that the ball moves in an elliptical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b \cos \omega t + \mathbf{j}2b \sin \omega t$$

where b and ω are constants. Find the speed of the ball as a function of t. In particular, find v at $t = 0$ and at $t = \pi/2\omega$, at which times the ball is, respectively, at its minimum and maximum distances from the origin.

1.18 A buzzing fly moves in a helical path given by the equation

$$\mathbf{r}(t) = \mathbf{i}b \sin \omega t + \mathbf{j}b \cos \omega t + \mathbf{k}ct^2$$

Show that the magnitude of the acceleration of the fly is constant, provided b, ω, and c are constant.

1.19 A bee goes out from its hive in a spiral path given in plane polar coordinates by

$$r = be^{kt} \qquad \theta = ct$$

where b, k, and c are positive constants. Show that the angle between the velocity vector and the acceleration vector remains constant as the bee moves outward. (*Hint: Find* $\mathbf{v} \cdot \mathbf{a}/va$.)

1.20 Work Problem 1.18 using cylindrical coordinates where $R = b$, $\phi = \omega t$, and $z = ct^2$.

1.21 The position of a particle as a function of time is given by

$$\mathbf{r}(t) = \mathbf{i}(1 - e^{-kt}) + \mathbf{j}e^{kt}$$

where k is a positive constant. Find the velocity and acceleration of the particle. Sketch its trajectory.

1.22 An ant crawls on the surface of a ball of radius b in such a manner that the ant's motion is given in spherical coordinates by the equations

$$r = b \qquad \phi = \omega t \qquad \theta = \frac{\pi}{2}\left[1 + \frac{1}{4}\cos(4\omega t)\right]$$

Find the speed of the ant as a function of the time t. What sort of path is represented by the above equations?

1.23 Prove that $\mathbf{v} \cdot \mathbf{a} = v\dot{v}$ and, hence, that for a moving particle $\mathbf{v}$ and $\mathbf{a}$ are perpendicular to each other if the speed v is constant. (*Hint: Differentiate both sides of the equation* $\mathbf{v} \cdot \mathbf{v} = v^2$ *with respect to* t. *Note,* $\dot{v}$ *is not the same as* $|\mathbf{a}|$. *It is the magnitude of the acceleration of the particle along its instantaneous direction of motion.*)

1.24 Prove that

$$\frac{d}{dt}[\mathbf{r} \cdot (\mathbf{v} \times \mathbf{a})] = \mathbf{r} \cdot (\mathbf{v} \times \dot{\mathbf{a}})$$

1.25 Show that the tangential component of the acceleration of a moving particle is given by the expression

$$a_\tau = \frac{\mathbf{v} \cdot \mathbf{a}}{v}$$

and the normal component is therefore

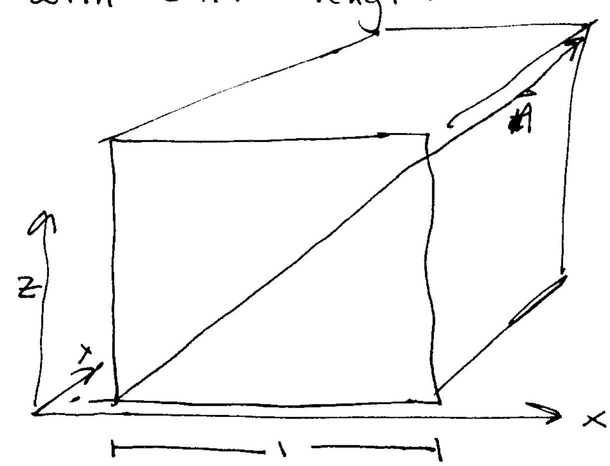
$$a_n = (a^2 - a_\tau^2)^{1/2} = \left[a^2 - \frac{(\mathbf{v} \cdot \mathbf{a})^2}{v^2}\right]^{1/2}$$

1.26 Use the above result to find the tangential and normal components of the acceleration as functions of time in Problems 1.18 and 1.19.

1.27 Prove that $|\mathbf{v} \times \mathbf{a}| = v^3/\rho$, where ρ is the radius of curvature of the path of a moving particle.

1.28 A wheel of radius b rolls along the ground with constant forward acceleration a_0. Show that, at any given instant, the magnitude of the acceleration of any point on the wheel is $(a_0^2 + v^4/b^2)^{1/2}$ relative to the center of the wheel and is also $a_0[2 + 2\cos\theta + v^4/a_0^2b^2 - (2v^2/a_0b)\sin\theta]^{1/2}$ relative to the ground. Here v is the instantaneous forward speed, and θ defines the location of the point on the wheel, measured forward from the highest point. Which point has the greatest acceleration relative to the ground?

Problem 1.4 Cube with unit length



(a) Vector along major diagonal

$$\vec{A} = (1, 1, 1)$$

(b) Vector along diagonal of bottom face

$$\vec{B} = (1, 1, 0)$$

(c) $$\vec{A} \times \vec{B} = \begin{pmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= -1\,\hat{i} + \hat{j} = -\hat{i} + \hat{j}$$

(d) $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta$

$$\theta = \sin^{-1}\left(\frac{|\vec{A} \times \vec{B}|}{|\vec{A}|\,|\vec{B}|}\right) = \sin^{-1}\left(\frac{\sqrt{1^2+1^2}}{\left(\sqrt{1^2+1^2+1^2}\right)\left(\sqrt{1^2+1^2}\right)}\right)$$

$$= \sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}\,\sqrt{2}}\right) = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\theta = 35.26^\circ$$

Problem 1.14

Rotate the vector $\vec{v} = (2, 3, -1)$ by $30°$ about the z axis.

The transformation matrix is

$$T = \begin{pmatrix} \cos 30° & \sin 30° & 0 \\ -\sin 30° & \cos 30° & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.867 & 0.5 & 0 \\ -0.5 & 0.867 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiply the transformation matrix by the vector.

$$\vec{v}\,' = T\vec{v} = \begin{pmatrix} 0.867 & 0.5 & 0 \\ -0.5 & 0.867 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$$

$$= (3.23, 1.6, -1)$$

Problem 1.16 - Car moves in circle of radius b

$$\vec{v}(t) = ct\, e_\theta$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = c e_\theta + ct \frac{de_\theta}{dt}$$

$$= c e_\theta - ct\dot{\theta} e_r$$

$$\theta = \frac{s}{b} \qquad \dot{\theta} = \frac{\dot{s}}{b} = \frac{v}{b} = \frac{ct}{b}$$

$$\vec{a}(t) = c e_\theta - \frac{c^2 t^2}{b} e_r$$

Use hint

Normal component $-\dfrac{c^2 t^2}{b}$

Tangential component c

$$\frac{c^2 t^2}{b} = c$$

$$t^2 = \frac{b}{c}$$

$$t = \sqrt{b/c}$$

Problem 1.18

Fly travels in helical path

$$\vec{r}(t) = \hat{i}\, b \sin\omega t + \hat{j}\, b\cos\omega t + \hat{k}\, ct^2$$

$$\vec{v}(t) = \hat{i}\,\omega b\cos\omega t - \hat{j}\, b\omega \sin\omega t + \hat{k}\, 2ct$$

$$\vec{a}(t) = -\hat{i}\,\omega^2 b\sin\omega t - \hat{j}\, b\omega^2\cos\omega t + \hat{k}\, 2c$$

$$a = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{(\omega^2 b\sin\omega t)^2 + (b\omega^2\cos\omega t)^2 + 4c^2}$$

$$= \sqrt{\omega^4 b^2(\sin^2\omega t + \cos^2\omega t) + 4c^2}$$

$$= \sqrt{4c^2 + \omega^4 b^2}$$

Factor be^{kt} out of everything

$$\vec{v} = be^{kt}\left(k e_r + c\, e_\theta\right)$$

$$\vec{a} = be^{kt}\left[(k^2 - c^2)\, e_r + 2kc\, e_\theta\right]$$

At this point we can recognize the time dependence will cancel out.

Carry it through

$$\vec{v}\cdot\vec{a} = b^2 e^{2kt}\left[k(k^2-c^2) + 2kc^2\right]$$

$$|\vec{v}| = be^{kt}\sqrt{k^2+c^2}$$

$$|\vec{a}| = be^{kt}\sqrt{(k^2-c^2)^2 + 4k^2c^2}$$

$$\cos\theta = \frac{\vec{v}\cdot\vec{a}}{|\vec{v}||\vec{a}|} = \frac{k(k^2-c^2) + 2kc^2}{\left(\sqrt{k^2+c^2}\right)\left(\sqrt{(k^2-c^2)^2+4k^2c^2}\right)}$$

Does not depend on time

$$\cos\theta = \frac{k}{\sqrt{k^2+c^2}}$$

Problem 1.20

In polar coordinates, a fly travels in a trajectory

$$R = b \quad , \quad \phi = \omega t \quad , \quad z = ct^2$$

$$\vec{a}(t) = (\ddot{r} - r\dot{\theta}^2)e_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})e_\theta + \ddot{z}e_z$$

$$= -b\omega^2 e_r + 2c\, e_z$$

$$|\vec{a}| = \sqrt{b^2\omega^4 + 4c^2}$$

Problem 1.22

Ant moves in spherical coordinates

$$r = b \quad , \quad \phi = \omega t, \quad \theta = \frac{\pi}{2}\left[1 + \frac{1}{4}\cos 4\omega t\right]$$

$$\vec{v}(t) = \dot{r}\, e_r + r\dot{\phi}\sin\theta\, e_\phi + e_\theta\, r\dot{\theta}$$

$$\dot{r} = 0 \qquad \dot{\phi} = \omega \qquad \dot{\theta} = -\frac{\pi\omega}{2}\sin 4\omega t$$

$$\vec{v}(t) = \cancel{b\omega \sin \frac{\pi}{2}\left[1 + \frac{1}{4}\cos 4\omega t\right] e_\phi}$$

$$= b\omega \sin \frac{\pi}{2}\left[1 + \frac{1}{4}\cos 4\omega t\right] e_\phi$$

$$- \frac{b\pi\omega}{2}\sin 4\omega t\; e_\theta$$

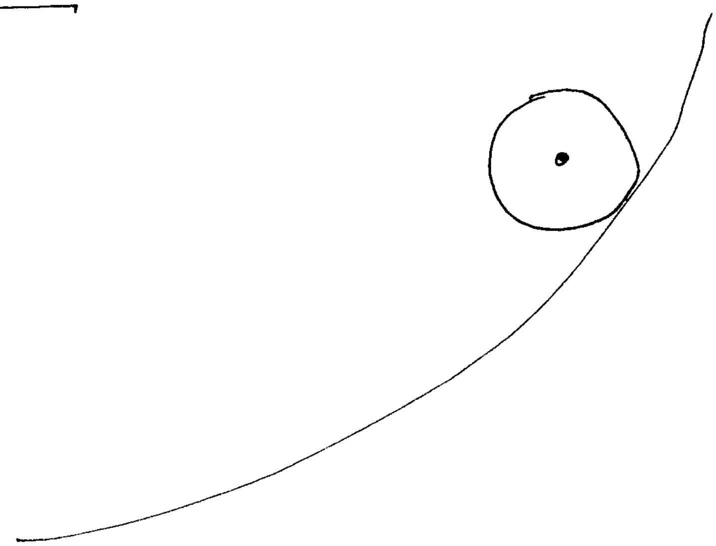
The trajectory is the ant walking down ~~the ball~~ along the equator and staggering.

Problem 1.23

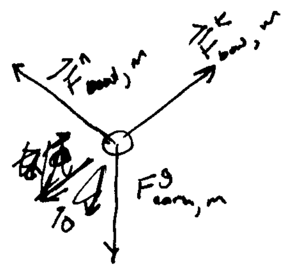
$$\vec{v} \cdot \vec{v} = v^2$$

$$\frac{d}{dt}\left(\vec{v} \cdot \vec{v}\right) = 2\,\vec{v} \cdot \vec{a} = 2\,v\,\dot{v}$$

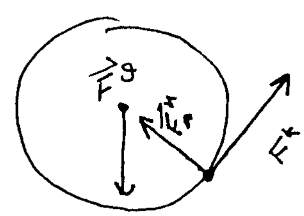
Review Problem 1



Free Body



Extended Free Body



R2 Select the stake (initial location) as the axis about which angular momentum is evaluated.

~~Three~~ Interactions - I Collision II Rotation

Universe as System -

Momentum, Energy, and Angular Momentum Conserved.

Before Collision $p_1 = mv$ $p_2 = 0$ $p_{earth} = 0$

After Collision $p_1 = 0$ $p_2 = mv$ $p_{earth} = 0$

$\Rightarrow$ Momentum and energy conserved.
$\Rightarrow$ Angular momentum conserved about any axis in a linear collision.

Sln - Elastic so $T_i = T_f$
$|v_{rel,i}| = |v_{ref,f}|$

check that solution where on puck stops meets these conditions.

During Rotation - The puck orbits in a circular orbit about the center of mass of the earth puck system.

Approximation I - Earth fixed
In this case, the puck moves in a ~~fix~~ circular orbit at ~~fix~~ constant energy. The pucks momentum is constantly changing so this approximation does not conserve momentum.

Approximation II — Earth point particle at stake.

The puck earth system rotates about its center of mass. Angular momentum is conserved, since the force is central.

The earth and puck exchange momentum and energy during the rotation.

Approximation III — Earth sphere. The central force of the string exerts a torque on the earth. The puck/earth system exchange momentum, energy, and angular ~~momentum~~ momentum.

(b) Puck as system

Before Interaction Collision —

$p=0$, $T=0$, $L=0$

Collision — Impulse and work and torque done to puck

$p = 0 + \text{impulse}$, $T = 0 + W$,

$L = 0 + \text{Angular impulse.}$

During Rotation — (Approx II)

Impulse, Work, done on puck no torque, so angular momentum remains fixed.

B1 The classic blunders (Princess Bride)

(1) Never get involved in a land war in Asia

(2) Don't go up against a Sicilian when death is on the line.