Mechanics Spring 2003 - Homework 11

Due Never - Solution in Library

Problem 11.1 Work Fowles Problem 11.2

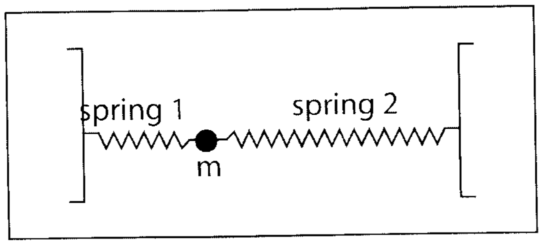
Problem 11.2 The system below is composed of one mass m and two springs with spring constants $k_1 = 2$ and $k_2 = 3$. The springs have un-stretched lengths $l_1 = l_2 \equiv \ell_0$. The supports to which the springs are attached are $d = 3\ell_0$ apart. Let x be the distance the mass is from the left support.

(a) Write the potential energy function for the system.

(b) Find the equilibrium location of the mass from the potential.

(c) Mathematically show the equilibrium is stable.

(d) Find the frequency of oscillation of the system.



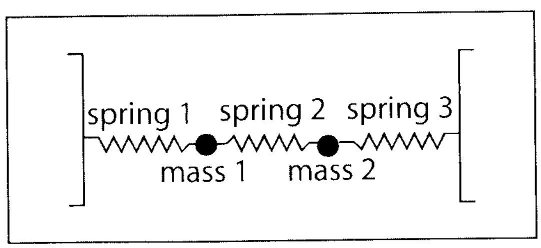
Problem 11.3 The system below has two masses each of mass m and three springs with spring constants $k_1 = 1$, $k_2 = 2$, and $k_3 = 3$. Let the equilibrium location of the masses be given by x_1 and x_2, measured from the left support. The masses oscillate in a line.

(a) Write the Lagrangian for the system.

(b) Find the frequencies of the normal modes of the system.

(c) Find the eigenvectors of the normal modes.

(d) How must the system be prepared for it to oscillate only with the lower normal frequency?



Problem 11.4 Work Fowles Problem 11.20. Hint: Find the Lagrangian and make small oscillation approximation. The $\cos\theta$ term in the kinetic energy becomes 1 if θ is small. You only have to find the normal frequencies.

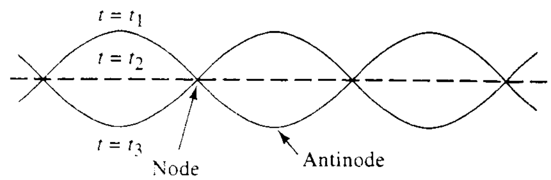


Figure 11.6.3 A standing sinusoidal wave.

facts are illustrated in Figure 11.6.3. Note again that there is a well-defined constraint on the values of allowable wavelengths λ. Since the endpoints of the string are fixed, we have as boundary conditions

$$q = 0 \qquad (x = 0, L) \tag{11.6.15}$$

that our solution (Equation 11.6.14) must obey. The first condition at $x = 0$ is met automatically. The second boundary condition at $x = L$ is met if

$$L = N\left(\frac{\lambda}{2}\right) \qquad \lambda = \frac{2L}{N} \tag{11.6.16}$$

An integral number of half wavelengths must fit within the length L if the endpoints are to be nodes. This is precisely the condition obtained previously for the normal modes of the loaded string.

PROBLEMS

11.1 A particle of mass m moves in one-dimensional motion with the following potential energy functions:

(a) $V(x) = \dfrac{k}{2}x^2 + \dfrac{k^2}{x}$

(b) $V(x) = kxe^{-bx}$

(c) $V(x) = k(x^4 - b^2x^2)$

where all constants are real and positive. Find the equilibrium positions for each case and determine their stability.

(d) Find the angular frequency ω for small oscillations about the respective positions of stable equilibrium for parts (a), (b), and (c), and find the period in seconds for each case if $m = 1$ g, and k and b are each of unit value in cgs units.

11.2 A particle moves in two dimensions under the potential energy function

$$V(x, y) = k(x^2 + y^2 - 2bx - 4by)$$

where k is a positive constant. Show that there is one position of equilibrium. Is it stable or unstable?

11.3 The potential energy function of a particle of mass m in one-dimensional motion is given by

$$V(x) = -\frac{k}{2}x^2$$

and so the force is of the antirestoring type

$$F(x) = kx$$

with $x = 0$ as a position of unstable equilibrium when k is a positive constant. If the initial conditions are $t = 0$, $x = x_0$, and $\dot{x} = 0$, show that the ensuing motion is given by an exponential "runaway"

$$x(t) = x_0(e^{\alpha t} + e^{-\alpha t})/2$$

where the constant $\alpha = \sqrt{k/m}$.

11.4 A light elastic cord of length $2l$ and stiffness k is held with the ends fixed a distance $2l$ apart in a horizontal position. A block of mass m is then suspended from the midpoint of the cord. Show that the potential energy of the system is given by the expression

$$V(y) = 2k[y^2 - 2l(y^2 + l^2)^{1/2}] - mgy$$

where y is the vertical sag of the center of the cord. From this show that the equilibrium position is given by a root of the equation

$$u^4 - 2au^3 + a^2u^2 - 2au + a^2 = 0$$

where $u = y/l$ and $a = mg/4kl$.

11.5 A uniform cubical block of mass m and sides $2a$ is balanced on top of a rough sphere of radius b. Show that the potential energy function can be expressed as

$$V(\theta) = mg[(a + b)\cos\theta + b\theta\sin\theta]$$

where θ is the angle of tilt. From this, show that the equilibrium at $\theta = 0$ is stable, or unstable, depending on whether a is less than or greater than b, respectively.

11.6 Expand the potential energy function of Problem 11.5 as a power series in θ. From this determine the stability for the case $a = b$.

11.7 A solid homogeneous hemisphere of radius a rests on top of a rough hemispherical cap of radius b, the curved faces being in contact. Show that the equilibrium is stable if a is less than $3b/5$.

11.8 Determine the frequency of vertical oscillations about the equilibrium position in Problem 11.4.

11.9 Determine the period of oscillation of the block in Problem 11.5.

11.10 Determine the period of oscillation of the hemisphere in Problem 11.7.

11.11 A small steel ball rolls back and forth about its equilibrium position in a rough spherical bowl. Show that the period of oscillation is $2\pi[7(b - a)/5g]^{1/2}$, where a is the radius of the ball and b is the radius of the bowl. Find the period in seconds if $b = 1$ m and $a = 1$ cm.

11.12 For an orbiting satellite in the form of a thin rod, show that the stable equilibrium attitude and period of oscillation are the same as those found in Example 11.2.2 for the dumbbell satellite.

11.13 In the system of two identical coupled oscillators shown in Figure 11.3.1, one oscillator is started with initial amplitude A_0, whereas the other is at rest at its equilibrium position, so that the initial conditions are

$$t = 0 \qquad x_1(0) = A_0 \qquad x_2(0) = 0 \qquad \dot{x}_1(0) = \dot{x}_2(0) = 0$$

Show that the amplitude of the symmetric component is equal to the amplitude of the antisymmetric component in this case and that the complete solution can be expressed as follows:

$$x_1(t) = \frac{1}{2}A_0(\cos\omega_a t + \cos\omega_b t) = A_0\cos\overline{\omega}t\cos\Delta t$$

$$x_2(t) = \frac{1}{2}A_0(\cos\omega_a t - \cos\omega_b t) = A_0\sin\overline{\omega}t\sin\Delta t$$

in which $\overline{\omega} = (\omega_a + \omega_b)/2$ and $\Delta = (\omega_b - \omega_a)/2$. Thus, if the coupling is very weak so that $K' \ll K$, then $\overline{\omega}$ will be very nearly equal to $\omega_a = (K/m)^{1/2}$, and Δ is very small. Consequently, under the stated initial conditions, the first oscillator will eventually come to rest while the second oscillator oscillates with amplitude A_0. Later, the system will return to the initial condition, and so on. Thus, the energy passes back and forth between the two oscillators indefinitely.

11.14 In Problem 11.13 show that, for weak coupling, the period at which the energy trades back and forth is approximately equal to $T_a(K/2K')$ where $T_a = 2\pi/\omega_a = 2\pi/(m/K)^{1/2}$ is the period of the symmetric oscillation.

11.15 Two identical simple pendulums are coupled together by a very weak force of attraction that varies as the inverse square of the distance between the two particles. (This force might be the gravitational attraction between the two particles, for instance.) Show that, for small departures from the equilibrium configuration, the Lagrangian can be reduced to the same mathematical form, with appropriate constants, as that of the two identical coupled oscillators treated in Section 11.3 and in Problem 11.13. (*Hint: Consider Equation 11.3.9.*)

11.16 Find the normal frequencies of the coupled harmonic oscillator system (see Figure 11.3.1) for the general case in which the two particles have unequal mass and the springs have different stiffness. In particular, find the frequencies for the case $m_1 = m$, $m_2 = 2m$, $K_1 = K$, $K_2 = 2K$, $K' = 2K$. Express the result in terms of the quantity $\omega_0 = (K/m)^{1/2}$.

11.17 A light elastic spring of stiffness K is clamped at its upper end and supports a particle of mass m at its lower end. A second spring of stiffness K is fastened to the particle and, in turn, supports a particle of mass $2m$ at its lower end. Find the normal frequencies of the system for vertical oscillations about the equilibrium configuration. Find also the normal coordinates.

11.18 Consider the case of a double pendulum, Figure 11.3.7a, in which the two sections are of different length, the upper one being of length l_1 and the lower of length l_2. Both particles are of equal mass m. Find the normal frequencies of the system and the normal coordinates.

11.19 Set up the secular equation for the case of three coupled particles in a linear array and show that the normal frequencies are the same as those given by Equation 11.5.17.

11.20 A simple pendulum of mass m and length a is attached to a block of mass M that is constrained to slide along a frictionless, horizontal track as shown in Figure P11.20. Find the normal frequencies and normal modes of oscillation.

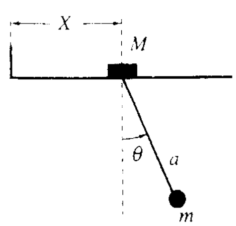


Figure P11.20

11.21 Three beads of mass m, m, and $2m$ are constrained to slide along a frictionless, circular hoop. The two small masses are each connected to the large mass and also to each other by springs of length a and force constants k and k', respectively. The masses are shown in Figure P11.21 at their equilibrium positions, which are located at 120° angular separations. The largest mass is initially displaced 10° clockwise from its equilibrium position, and the other two are held fixed in place. The three masses are then simultaneously released from rest.

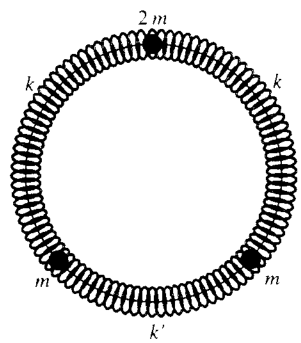


Figure P11.21

(a) Find the normal frequencies and normal modes of oscillation.

(b) Solve for the resulting motion of each mass.

11.22 Find the matrix **A** that diagonalizes the **K** and **M** matrices in the case of the linear triatomic molecule of Example 11.4.1. Show that the ratio of their diagonal elements is equal to the eigenfrequencies of the normal modes of oscillation.

11.23 A triatomic molecule like hydrogen sulfide (H_2S) consists of two hydrogen atoms of mass m and one sulfur atom of mass M constrained by atomic bonding forces to assume the triangular configuration shown in Figure P11.23. Assume that the bonding forces can be approximated by springs whose force constant is k. When the three atoms are in their equilibrium configuration, the HS distance is $a = 1.67 \times 10^{-10}$ m and the H — S — H vertex angle is approximately $2\alpha = 90°$. Find the normal frequencies and normal modes of oscillation. Assume that the hydrogen atoms do not interact directly with each other.

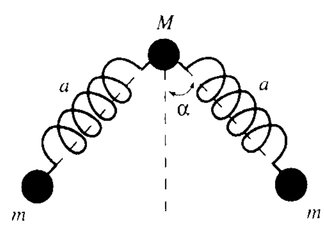


Figure P11.23

11.24 Two waves are traveling through a medium. Assume that the displacements from equilibrium of particles that make up the medium are given by the functions

$$q_1(x, t) = Ae^{i(\omega t - kx)}$$
$$q_2(x, t) = Ae^{i(\Omega t - Kx)}$$

whose real part represents the physical wave.

(a) Show that each of these functions are solutions of the wave equation.

(b) Assume that the frequencies and wave numbers differ by small amounts

$$\Omega = \omega + \Delta\omega \qquad K = k + \Delta k$$

Ignoring small differences of second order, show that the real part of the resultant wave function is given approximately by

$$Q(x, t) = q_1 + q_2 \approx 2 \cos\left[\frac{(\Delta\omega)t - (\Delta k)x}{2}\right] \cos(\omega t - kx)$$

The resultant wave has the same frequency and wave number as the original wave, but it has a modulated amplitude (the wave number $k = 2\pi/\lambda$).

(c) Calculate the speed of propagation of the amplitude modulation (this speed is called the group speed v_g of the wave).

11.25 Illustrate the normal modes for the case of four particles in a linear array. Find the numerical values of the ratios of the second, third, and fourth normal frequencies to the lowest or first normal frequency.

11.26 A light elastic cord of natural length l and stiffness K is stretched out to a length $l + \Delta l$ and loaded with a number n of particles evenly spaced along its length. If m is the total mass of all n particles, find the speed of transverse and of longitudinal waves in the cord.

11.27 Work Problem 11.26 for the case in which, instead of being loaded, the cord is heavy with linear mass density μ.

COMPUTER PROBLEMS

C 11.1 Consider a single pulse traveling down an infinitely long string. Assume that at $t = 0$, the shape of the pulse, or the vertical displacement of the string, is

$$y(x) = \frac{1}{1 + x^2} \tag{1}$$

Analogous to the discussion of Fourier series in Section 3.7, this pulse can be thought of as a superposition of harmonic waves of differing wave numbers k. The infinite sum of Section 3.7, however, that approximates a repetitive function needs to be replaced here by an integral over an infinite number of harmonic waves, each one weighted by an appropriate amplitude function, that is,

$$y(x) = \int_0^\infty a(k) \cos(kx)\, dk \tag{2}$$

We use cosine functions since $y(x)$ is an even function of x. The amplitude function $a(k)$ is given by

$$a(k) = \frac{2}{\pi} \int_0^\infty y(x) \cos(kx)\, dx \tag{3}$$

(a) Calculate $a(k)$ using Equation 3.

(b) Substitute $a(k)$ into Equation 2 and show that it yields $y(x)$.

7

Fowles 11.2

$$V(x,y) = k(x^2+y^2-2bx-4by)$$

$$\frac{\partial V}{\partial x} = 2kx - 2bk = 0$$

$$\Rightarrow x = b$$

$$\frac{\partial V}{\partial y} = 2ky - 4bk = 0$$

$$y = 2b$$

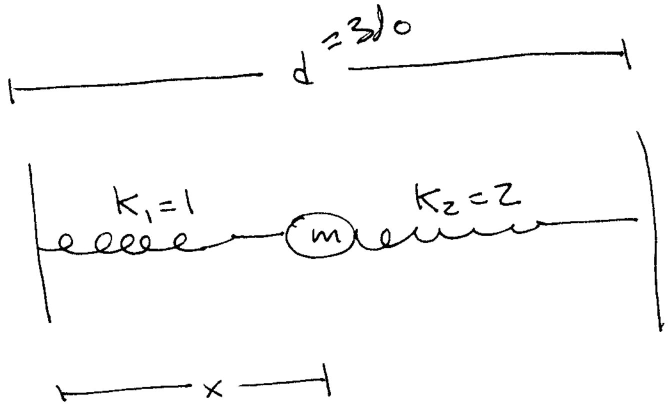
$$K_{11} = \left.\frac{\partial^2 V}{\partial x^2}\right|_{2,4} = 2k \qquad K_{12} = K_{21} = \frac{\partial^2 V}{\partial x \partial y} = 0$$

$$K_{22} = \left.\frac{\partial^2 V}{\partial y^2}\right|_{2,4} = 2k$$

$$\begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} = \begin{vmatrix} 2k & 0 \\ 0 & 2k \end{vmatrix} = 4k^2 > 0$$

so stable.

2



$$V = \frac{1}{2} k_1 (x - l_0)^2 + \tfrac{1}{2} k_2 (d - x - l_0)^2$$

$$\frac{\partial V}{\partial x} = \cancel{2} k_1 (x - l_0) - k_2 (2l_0 - x) = 0$$

$$= x - l_0 - 4l_0 + 2x = 0$$

$$\cancel{x = 3l_0}$$

$$x = \frac{5l_0}{3}$$

Check for stability

Stable if $\frac{\partial^2 V}{\partial x^2} > 0$

$$\frac{\partial^2 V}{\partial x^2} = 3 > 0 \quad \text{so stable}$$

(c) Linearized Lagrangian

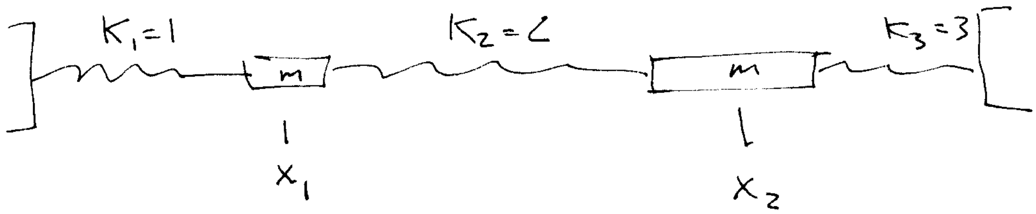
$$\frac{1}{2}m\dot{x}^2 - \frac{1}{2}\left(\frac{\partial^2 V}{\partial x^2}\right)_{eq}(x-x_0)^2 = 0 \qquad x_0 = \tfrac{5}{3}l_0$$

<u>EOM</u>

$$m\ddot{x} + \left(\frac{\partial^2 V}{\partial x^2}\right)(x-x_0) = 0$$

$$\omega = \sqrt{\frac{\partial^2 V/\partial x^2\big|_{5/3 l_0}}{m}} = \sqrt{3/m}$$

Example



$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_3 x_2^2 + \underbrace{\frac{1}{2} k_2 (x_2 - x_1)^2}_{\frac{1}{2} k_2 (x_1^2 - 2x_1x_2 + x_2^2)}$$

$$K_{11} = \frac{\partial^2 V}{\partial x_1 \partial x_1} = k_1 + k_2$$

$$K_{22} = \frac{\partial^2 V}{\partial x_2 \partial x_2} = k_3 + k_2$$

$$K_{12} = K_{21} = \frac{\partial^2 V}{\partial x_1 \partial x_2} = -k_2$$

$$M_{11} = m \qquad M_{22} = M \qquad M_{12} = M_{21} = 0$$

$$\overleftrightarrow{M}\ddot{q} + \overleftrightarrow{K}q = 0$$

$$\overleftrightarrow{K}\vec{q} = \omega^2 \overleftrightarrow{M}\vec{q}$$

$$\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} k_1+k_2-m\omega^2 & -k_2 \\ -k_2 & k_2+k_3-m\omega^2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Has solution if $\det\left|\left(\ \right)\right| = 0$

$$\det\begin{vmatrix} 3-m\omega^2 & -2 \\ -2 & 5-m\omega^2 \end{vmatrix} = 0$$

$$(3-m\omega^2)(5-m\omega^2) - 4 = 0$$

$$15 - 8m\omega^2 + (m\omega^2)^2 - 4 = 0$$

$$(m\omega^2)^2 - 8m\omega^2 + 11 = 0$$

$$m\omega^2 = \frac{8 \pm \sqrt{64-44}}{2} = 4 \pm \sqrt{5}$$

Construct normal mode for $m\omega_1^2 = 4+\sqrt{5}$

$$\begin{pmatrix} 3-(4+\sqrt{5}) & -2 \\ -2 & 3-(4+\sqrt{5}) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Let $x_1 = 1$

$$3-(4+\sqrt{5}) \quad -2x_2 = 0$$

$$x_2 = \frac{-1-\sqrt{5}}{2}$$

$$\vec{a}_1 = \begin{pmatrix} 1 \\ -\frac{(1+\sqrt{5})}{2} \end{pmatrix}$$

$$\vec{X}_1 = \begin{pmatrix} 1 \\ -\frac{(1+\sqrt{5})}{2} \end{pmatrix} \cos\left(\sqrt{\frac{4+\sqrt{5}}{m}}\, t + \sigma \right)$$

Construct Normal Mode for $m\omega_2^2 = 4-\sqrt{5}$

$$\begin{pmatrix} 3-(4-\sqrt{5}) & -2 \\ -2 & 3-(4-\sqrt{5}) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

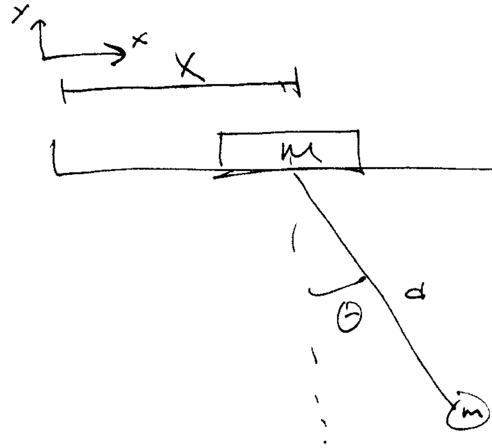
Let $x_1 = 1$

$$-1+\sqrt{5} \quad -2x_2 = 0$$

$$x_2 = \frac{-1+\sqrt{5}}{2}$$

$$\vec{x}_2 = \begin{pmatrix} 1 \\ \frac{-1+\sqrt{2}}{2} \end{pmatrix} \cos\left(\sqrt{\frac{4-\sqrt{5}}{m}}\, t + \delta\right)$$

Problem 4



$$T = \frac{1}{2} M \dot{X}^2 + T_{pend}$$

$$x = X + a \sin\theta$$

$$y = -a\cos\theta$$

$$\dot{x} = \dot{X} + a\cos\theta\,\dot{\theta}$$

$$\dot{y} = +a\sin\theta\,\dot{\theta}$$

$$\dot{x}^2 + \dot{y}^2 = \dot{X}^2 + a^2\dot{\theta}^2 + 2a\dot{X}\cos\theta\,\dot{\theta}$$

For small oscillation, $\cos\theta \sim 1$ $\left(\cos\theta = 1 + \frac{1}{2}\theta^2 \ldots\right)$

$$L = T - V = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m \dot{X}^2 + \frac{1}{2} m a^2 \dot{\theta}^2 + m a \dot{X}\dot{\theta}$$

$$+ m g a \cos\theta = 0$$

$$\frac{\partial V}{\partial \theta} = + m g a \sin\theta = 0 \Rightarrow \theta = 0$$

$$\frac{\partial^2 V}{\partial \theta^2} = m g a \cos\theta \Big|_{\theta=0} = m g a$$

$$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ma^2\dot{\theta}^2 + ma\dot{x}\dot{\theta} - \frac{1}{2}mga\theta^2 = 0$$

x eqn

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 - \frac{d}{dt}\left((M+m)\dot{x} + ma\dot{\theta}\right)$$

$$= (M+m)\ddot{x} + ma\ddot{\theta} = 0$$

θ eqn

$$-mga\theta - \frac{d}{dt}\left(ma^2\dot{\theta} + ma\dot{x}\right) = 0$$

$$-mga\theta - ma^2\ddot{\theta} - ma\ddot{x} = 0$$

$$\ddot{x} + a\ddot{\theta} + g\theta = 0$$

$$\overleftrightarrow{M}\ddot{\vec{q}} + \overleftrightarrow{K}\vec{q} = 0 \leftarrow \cancel{\omega^2 M q}$$

$$\begin{pmatrix} a & 1 \\ ma & M+m \end{pmatrix}\begin{pmatrix} \ddot{\theta} \\ \ddot{x} \end{pmatrix} + \begin{pmatrix} +g & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \theta \\ x \end{pmatrix} = 0$$

<u>Guess</u>

$$\Theta(t) = \Theta_0 \cos(\omega t + \delta)$$

$$X(t) = X_0 \cos(\omega t + \delta)$$

$$\begin{pmatrix} \ddot{\Theta} \\ \ddot{X} \end{pmatrix} = -\omega^2 \begin{pmatrix} \Theta_0 \\ X_0 \end{pmatrix} \cos(\omega t + \delta)$$

$$\begin{pmatrix} -a\omega^2 & -\omega^2 \\ -ma\omega^2 & -\omega^2(M+m) \end{pmatrix} \begin{pmatrix} \Theta \\ X \end{pmatrix} + \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Theta \\ X \end{pmatrix} = 0$$

$$\begin{pmatrix} g - a\omega^2 & -\omega^2 \\ -ma\omega^2 & -\omega^2(M+m) \end{pmatrix} \begin{pmatrix} \Theta \\ X \end{pmatrix} = 0$$

$$\det \begin{vmatrix} g - a\omega^2 & -\omega^2 \\ -ma\omega^2 & -\omega^2(M+m) \end{vmatrix} = 0$$

$$-\omega^2 (M+m)(g - a\omega^2) - ma\omega^4 = 0$$

$\omega^2 = 0$ is a root.

$\cancel{g = a\omega^2}$

$$g(M+m) - a(M+m)\omega^2 + ma\omega^2 = 0$$

$$g(M+m) - aM\omega^2 = 0$$

$$\omega^2 = \frac{g(M+m)}{aM}$$

So normal frequencies are

$$\omega = 0 \qquad \omega = \sqrt{\frac{g(M+m)}{aM}}$$