Homework 4

Due Tuesday 2/18/2003

Fowles Problems

- 4.2 5 (2 each)
- 4.4 (3 each) 9
- 4.5 (3 ea) +6
- 4.6 10 pts
- 4.9 10 pts
- 4.14 8 pts
- 4.17 10 pts
- 4.18 5 pts

76 total 18

Problem E.1 Consider the force

3

$$\vec{F} = \frac{\hat{e}_\theta}{r}$$

expressed in spherical coordinates where θ is the angle that $\vec{r}$ makes with the z axis and r is the distance from the origin. Determine whether the force is conservative by evaluating the curl of the force in spherical coordinates.

4 pts

Problem E.2 An isotropic harmonic oscillator has potential function,

$$V(x, y) = \frac{1}{2}k(x^2 + y^2)$$

where k is the spring constant. The oscillator is confined to the $x - y$ surface. The mass experiencing the restoring force is m. The mass is released from the point $(a, a, 0)$ and brought to a stop at $(-b, -b, 0)$ where a and b are positive constants. The surface has a coefficient of kinetic friction of μ^k. How much work does the force that brings the particle to a stop do?

Problem E.3 My students in UPII routinely do math where the potential

7 pts

$$V(r) = \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

becomes the potential

$$V(r) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

In the $x - y$ plane, qualitatively sketch the contour map of the potential energy for each expression. Draw the lines of each potential on the same map in different colors. I am looking for qualitative difference in the shape of the two contours. Do not worry about using contours equally spaced in energy.

Bonus Problem B.1 If one wishes to play Global Thermonuclear War, what is the password?

Joshua

PROBLEMS

4.1 Find the force for each of the following potential energy functions:

(a) $V = cxyz + C$

(b) $V = \alpha x^2 + \beta y^2 + \gamma z^2 + C$

(c) $V = ce^{-(\alpha x + \beta y + \gamma z)}$

(d) $V = cr^n$ in spherical coordinates

4.2 By finding the curl, determine which of the following forces are conservative:

(a) $\mathbf{F} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$

(b) $\mathbf{F} = \mathbf{i}y - \mathbf{j}x + \mathbf{k}z^2$

(c) $\mathbf{F} = \mathbf{i}y + \mathbf{j}x + \mathbf{k}z^3$

(d) $\mathbf{F} = -kr^{-n}\mathbf{e}_r$ in spherical coordinates

4.3 Find the value of the constant c such that each of the following forces is conservative:

(a) $\mathbf{F} = \mathbf{i}xy + \mathbf{j}cx^2 + \mathbf{k}z^3$

(b) $\mathbf{F} = \mathbf{i}(z/y) + c\mathbf{j}(xz/y^2) + \mathbf{k}(x/y)$

4.4 A particle of mass m moving in three dimensions under the potential energy function $V(x, y, z) = \alpha x + \beta y^2 + \gamma z^3$ has speed v_0 when it passes through the origin.

(a) What will its speed be if and when it passes through the point (1, 1, 1)?

(b) If the point (1, 1, 1) is a turning point in the motion ($v = 0$), what is v_0?

(c) What are the component differential equations of motion of the particle?

(*Note:* It is *not* necessary to solve the differential equations of motion in this problem.)

4.5 Consider the two force functions

(a) $\mathbf{F} = \mathbf{i}x + \mathbf{j}y$

(b) $\mathbf{F} = \mathbf{i}y - \mathbf{j}x$

Verify that (a) is conservative and that (b) is nonconservative by showing that the integral $\int \mathbf{F} \cdot d\mathbf{r}$ is independent of the path of integration for (a), but not for (b), by taking two paths in which the starting point is the origin (0, 0), and the endpoint is (1, 1). For one path take the line $x = y$. For the other path take the x-axis out to the point (1, 0) and then the line $x = 1$ up to the point (1, 1).

4.6 Show that the variation of gravity with height can be accounted for approximately by the following potential energy function:

$$V = mgz\left(1 - \frac{z}{r_e}\right)$$

in which r_e is the radius of the Earth. Find the force given by the above potential function. From this find the component differential equations of motion of a projectile under such a force. If the vertical component of the initial velocity is v_{0z}, how high does the projectile go? (Compare with Example 2.3.2.)

4.7 Particles of mud are thrown from the rim of a rolling wheel. If the forward speed of the wheel is v_0, and the radius of the wheel is b, show that the greatest height above the ground that the mud can go is

$$b + \frac{v_0^2}{2g} + \frac{gb^2}{2v_0^2}$$

At what point on the rolling wheel does this mud leave?
(*Note:* It is necessary to assume that $v_0^2 \geq bg$.)

4.8 A gun is located at the bottom of a hill of constant slope ϕ. Show that the range of the gun measured up the slope of the hill is

$$\frac{2v_0^2 \cos \alpha \sin(\alpha - \phi)}{g \cos^2 \phi}$$

where α is the angle of elevation of the gun, and that the maximum value of the slope range is

$$\frac{v_0^2}{g(1 + \sin \phi)}$$

4.9 A baseball pitcher can throw a ball more easily horizontally than vertically. Assume that the pitcher's throwing speed varies with elevation angle approximately as $v_0 \cos \frac{1}{2}\theta_0$ m/s, where θ_0 is the initial elevation angle and v_0 is the initial velocity when the ball is thrown horizontally.
Find the angle θ_0 at which the ball must be thrown to achieve maximum **(a)** height and **(b)** range.
Find the values of the maximum **(c)** height and **(d)** range. Assume no air resistance and let $v_0 = 25$ m/s.

4.10 A gun can fire an artillery shell with a speed V_0 in any direction. Show that a shell can strike any target within the surface given by

$$g^2r^2 = V_0^4 - 2gV_0^2z$$

where z is the height of the target and r is its horizontal distance from the gun. Assume no air resistance.

4.11 Write down the component form of the differential equations of motion of a projectile if the air resistance is proportional to the square of the speed. Are the equations separated? Show that the x and y components of the velocity are given by

$$\dot{x} = \dot{x}_0 e^{-\gamma s} \qquad \dot{y} = \dot{y}_0 e^{-\gamma s}$$

where s is the distance the projectile has traveled along the path of motion, and $\gamma = c_2/m$.

4.12 Fill in the steps leading to Equations 4.3.18a and b, giving the horizontal range of a projectile that is subject to linear air drag.

4.13 The initial conditions for a two-dimensional isotropic oscillator are as follows: $t = 0$, $x = A$, $y = 4A$, $\dot{x} = 0$, $\dot{y} = 3\omega A$ where ω is the angular frequency. Find x and y as functions of t. Show that the motion takes place entirely within a rectangle of dimensions $2A$ and $10A$. Find the inclination ψ of the elliptical path relative to the x-axis. Make a sketch of the path.

4.14 A small lead ball of mass m is suspended by means of six light springs as shown in Figure 4.4.1. The stiffness constants are in the ratio 1:4:9, so that the potential energy function can be expressed as

$$V = \frac{k}{2}(x^2 + 4y^2 + 9z^2)$$

At time $t = 0$ the ball receives a push in the $(1, 1, 1)$ direction that imparts to it a speed v_0 at the origin. If $k = \pi^2 m$, numerically find x, y, and z as functions of the time t. Does the ball ever retrace its path? If so, for what value of t does it first return to the origin with the same velocity that it had at $t = 0$?

4.15 Complete the derivation of Equation 4.4.15.

4.16 An atom is situated in a simple cubic crystal lattice. If the potential energy of interaction between any two atoms is of the form $cr^{-\alpha}$, where c and α are constants and r is the distance between the two atoms, show that the total energy of interaction of a given atom with its six nearest neighbors is approximately that of the three-dimensional harmonic oscillator potential

$$V \simeq A + B(x^2 + y^2 + z^2)$$

where A and B are constants.
[*Note:* Assume that the six neighboring atoms are fixed and are located at the points $(\pm d, 0, 0)$, $(0, \pm d, 0)$, $(0, 0, \pm d)$, and that the displacement (x, y, z) of the given atom from the equilibrium position $(0, 0, 0)$ is small compared to d. Then $V = \Sigma\, cr_i^{-\alpha}$ where $r_1 = [(d - x)^2 + y^2 + z^2]^{1/2}$, with similar expressions for $r_2, r_3, \ldots, r_6$. See the approximation formulas in Appendix D.]

4.17 An electron moves in a force field due to a uniform electric field **E** and a uniform magnetic field **B** that is at right angles to **E**. Let $\mathbf{E} = \mathbf{j}E$ and $\mathbf{B} = \mathbf{k}B$. Take the initial position of the electron at the origin with initial velocity $\mathbf{v}_0 = \mathbf{i}v_0$ in the x direction. Find the resulting motion of the particle. Show that the path of motion is a cycloid:

$$x = a \sin \omega t + bt$$
$$y = a(1 - \cos \omega t)$$
$$z = 0$$

Cycloidal motion of electrons is used in an electronic tube called a magnetron to produce the microwaves in a microwave oven.

4.18 A particle is placed on a smooth sphere of radius b at a distance $b/2$ above the central plane. As the particle slides down the side of the sphere, at what point will it leave?

4.19 A bead slides on a smooth rigid wire bent into the form of a circular loop of radius b. If the plane of the loop is vertical, and if the bead starts from rest at a point that is level with the center of the loop, find the speed of the bead at the bottom and the reaction of the wire on the bead at that point.

4.20 Show that the period of the particle sliding in the cycloidal trough of Example 4.6.2 is $4\pi(A/g)^{1/2}$.

COMPUTER PROBLEMS

C 4.1 A bomber plane, about to drop a bomb, suffers a malfunction of its targeting computer. The pilot notes that there is a strong horizontal wind, so she decides to release the bomb anyway, directly over the visually sighted target, as the plane flies over it

4.2

(a) $\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$

conservative

(b) $\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & x^2 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) \; \hat{k}(-1-1)$

$= -2\hat{k}$

not-conservative

(c) $\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & z^3 \end{vmatrix} = 0\hat{i} + 0\hat{j} + (1-1)k = 0$

conservative

(d) $\nabla \times \vec{F} = \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & r\sin\theta\,\hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ -kr^{-n} & 0 & 0 \end{vmatrix} = 0$

conservative

4.4

$$E_{sys} = T(\vec{r},t) + V(\vec{r},t)$$

Given $T(\vec{r},0) = \frac{1}{2}mv_0^2$

$$E_{sys} = T(0,0) + V(0,0) = \text{const}$$

$$= \frac{1}{2}mv_0^2 + 0 = \frac{1}{2}mv_0^2$$

$$\frac{1}{2}mv_0^2 = T(\vec{r}_1) + V(\vec{r}_1)$$

$$= T(\vec{r}_1) + \alpha + \beta + \gamma$$

$$T(r_1) = \frac{1}{2}mv_0^2 - \alpha - \beta - \gamma = \frac{1}{2}mv_1^2$$

(a) $$\boxed{v_1 = \left(\frac{2}{m}\left(\frac{1}{2}mv_0^2 - \alpha - \beta - \gamma\right)\right)^{1/2}}$$

(b) If $v_1 = 0$,

$$\frac{1}{2}mv_0^2 = \alpha + \beta + \gamma$$

$$\boxed{v_0 = \left(\frac{2}{m}(\alpha + \beta + \gamma)\right)^{1/2}}$$

(c) $$m\frac{d\vec{v}}{dt} = F = -\nabla V = -\alpha\hat{i} - 2\beta y\hat{j} - 3\gamma z^2\hat{k}$$

<u>component EOM</u>

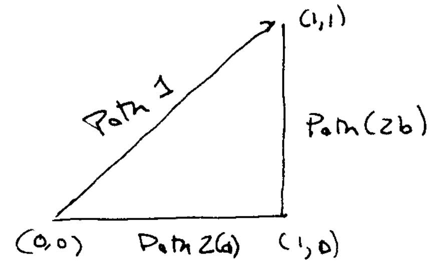
$\not{x}$ $$m\ddot{x} = -\alpha$$

$$m\ddot{y} = -2\beta y$$

$$m\ddot{z} = -3\gamma z^2$$

4.5 (Bonus)

(a) $\vec{F} = x\hat{i} + y\hat{j}$



For path 1, $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

$$W = \int \vec{F}\cdot d\vec{r} = \int_0^1 x\,dx + \int_0^1 y\,dy = 1$$

For path 2(a) $d\vec{r} = dx\,\hat{i}$ $y=0$

$$W = \int_0^1 \vec{F}\cdot d\vec{r} = \int_0^1 x\,dx = \frac{1}{2}$$ Path 2(a)

$d\vec{r} = dy\,\hat{j}$ $y=1$

$$W = \int_0^1 \vec{F}\cdot d\vec{r} = \int_0^1 y\,dy = 1/2$$ Path (2b)

Total Work Path 2 $= 1$ (Conservative)

(b) $\vec{F} = y\hat{i} - x\hat{j}$

Path 1 - $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$ $y = x$

$$W = \int \vec{F}\cdot d\vec{r} = \int_0^1 y\,dx - \int_0^1 x\,dy = 0$$

Path 2(a) $\qquad d\vec{r} = dx\,\hat{i} \qquad y = 0$

$$\int \vec{F} \cdot d\vec{r} = \int_0^1 y\,dx = \int 0\,dx = 0$$

Path 2(b) $\qquad d\vec{r} = dy\,\hat{j} \qquad x = 1$

$$\int \vec{F} \cdot d\vec{r} = -\int_0^1 x\,dy = -\int_0^1 dy = -1$$

$$W_{path\,1} = 0 \neq W_{path\,2} = -1$$

(4.6) The earth's potential is

$$V = -\frac{GMm}{r} + C$$

Select $V(r_e) = 0$ where r_e is the radius of the earth

$$V(r_e) = 0 \implies C = \frac{GmM}{r_e}$$

$$V = GMm\left[\frac{1}{r_e} - \frac{1}{r}\right] = GMm\left[\frac{1}{r_e} - \frac{1}{r_e+z}\right]$$

$$= \frac{GMm}{r_e}\left[1 - \frac{1}{1+z/r_e}\right]$$

Series Expansion

$$\frac{1}{1+x} = 1 - x + x^2 + \cdots$$

$$V = \frac{GMm}{r_e}\left[1 - \left(1 - \frac{z}{r_e} + \left(\frac{z}{r_e}\right)^2\right]\right.$$

$$V(r) = \frac{GMm}{r_e}\left[\frac{z}{r_e} - \left(\frac{z}{r_e}\right)^2\right]$$

$$= \frac{GMmz}{r_e^2}\left[1 - \frac{z}{r_e}\right]$$

$$g = \frac{GM}{r_e^2}$$

$$V(r) = mgz\left[1 - \frac{z}{r_e}\right]$$

$$F = -\nabla V(r) \qquad (\text{~~spherical~~ Cartesian})$$

$$= -\frac{\partial}{\partial z}\left[mgz - \frac{mgz^2}{r_e}\right]\hat{e}_z$$

$$= \left[-mg + \frac{2mgz}{r_e}\right]\hat{k}$$

Differential Eqn

$$m\ddot{z} = \left[-mg + \frac{2mgz}{r_e}\right]$$

$$\ddot{z} = -g + \frac{2g}{r_e}z = \frac{dv_z}{dt} = v\frac{dv}{dz}$$

Find velocity as a function of z, ~~wrong~~.

We have a potential function, find where $U(x_{max}) = T_0$

all the system energy is potential.

$$E_{sys} = U(x) + T(x) = T_0 + \underbrace{U(0)}_{0}$$

$$T_0 = mgz\left[1 - \frac{z}{r_e}\right]$$

$$\frac{z^2}{r_e} - z + \frac{T_0}{mg} = 0$$

$$z^2 - r_e z + \frac{T_0 r_e}{mg} = 0$$

Quadratic Eqn

$$z_{max} = \frac{r_e \pm \sqrt{r_e^2 - \frac{4 T_0 r_e}{mg}}}{2}$$

$$= \frac{r_e}{2}\left[1 \pm \sqrt{1 - \frac{4 T_0}{r_e m g}}\right]$$

select − root

$$= \frac{r_e}{2}\left[1 - \sqrt{1 - \frac{2 v_0^2}{r_e g}}\right]$$

(4.9) $\vec{v} = v\hat{v} = v\cos\theta\,\hat{\imath} + v\sin\theta\,\hat{j}$

$$v = v_0 \cos\left(\frac{\theta_0}{2}\right)$$

so at $\theta_0 = 90°$ $\quad v = v_0(0.7)$

Find Max Height h_{max} which occurs at time t_{max}.

$$v_y = v_{0y} - gt$$

At max height $v_y = 0$,

$$t_{max} = \frac{v_{0y}}{g}$$

The maximum height

$$h_{max} = v_{0y}\, t_{max} - \frac{1}{2} g t_{max}^2$$

$$\boxed{h_{max}(\theta) = \frac{v_{0y}^2}{g} - \frac{v_{0y}^2}{2g} = \frac{1}{2}\frac{v_{0y}^2}{g}}$$

$\cancel{h_{max} =}$ $\quad v_{0y} = v_0 \sin\theta = v_0 \sin\theta_0 \cos\frac{\theta_0}{2}$

$$h_{max} = \frac{v_0^2}{2g}\left(\sin\theta_0 \cos\frac{\theta_0}{2}\right)^2 = \frac{v_0}{2g}\sin^2\theta\cos^2\frac{\theta}{2}$$

Find the range, R, as a function of θ. The particle returns to the ground at time $t_f = 2\, t_{max}$.

$$X_{max} = X_{ox} t_f = 2\, t_{max} X_{ox} = \frac{2 v_0}{g} \cos^2\frac{\theta}{2} \cos\theta \sin\theta$$

Now maximize the height and range over θ

The angle for max height θ^h_{max} is given by

$$\frac{d h_{max}(\theta)}{d\theta} = \frac{2 v_0}{g}\left[+2 \sin\theta \cos\theta \cos^2\theta/2 - 2 \sin^2\theta \cos\theta/2 \sin\theta/2 \frac{1}{2}\right]$$

$$= \frac{2 v_0}{g} \underbrace{\left[2 \sin\theta \cos\theta \cos^2\frac{\theta}{2} - \sin^2\theta \cos\theta/2 \sin\theta/2\right] = 0}_{\text{Condition } \theta^h_{max} \text{ must meet.}}$$

<u>Solve for θ^h_{max} (optimal)</u>

Factor out $\sin\theta \cos\theta/2$

$$2 \cos\theta \cos\theta/2 - \sin\theta \sin\theta/2 = 0$$

Mistake introduce $\sqrt{}$

Go back and work with original

$$\cos\theta/2 \ \sin\theta/2 = \frac{1}{2}\sin\theta$$

$$\cos^2\theta/2 = \frac{1+\cos\theta}{2}$$

$$2\sin\theta\cos\theta\,\frac{(1+\cos\theta)}{2} - \sin^2\theta\,\frac{\sin\theta}{2} = 0$$

<u>Cancel $\sin\theta$</u>

$$\cos\theta + \cos^2\theta - \frac{1}{2}\sin^2\theta = 0$$

$$\cos\theta + \cos^2\theta - \frac{1}{2}(1-\cos^2\theta) = 0$$

$$3/2\cos^2\theta + \cos\theta - 1/2 = 0$$

$$\cos^2\theta + 2/3\cos\theta - 1/3 = 0$$

<u>Quadratic Eqn</u> $$\cos\theta = \frac{-2/3 \pm \sqrt{\frac{4}{9}+4/3}}{2} = -\cancel{4}/3 + \sqrt{4/9}$$

$$= -1/3 + 2/3 = 1/3$$

$$\theta = 70.5^0$$

Maximum Range R

$$R(\theta) = \frac{2v_0^2}{g} \cos^2\frac{\theta}{2} \cos\theta \sin\theta$$

Maximize

$$\frac{dR}{d\theta} = \frac{2v_0^2}{g}\left[-\cos\frac{\theta}{2} \sin\theta/2 \cos\theta\sin\theta - \cos^2\frac{\theta}{2} \sin^2\theta + \cos^2\frac{\theta}{2}\cos^2\theta \right] = 0$$

$[\;] = 0$ condition on θ^r_{max}

Solve θ^r_{max}

$$\cos^2\theta/2 = \frac{1+\cos\theta}{2}$$

$$\cos\frac{\theta}{2}\sin\frac{\theta}{2} = \frac{\sin\theta}{2}$$

$$-\frac{\sin\theta}{2}\cos\theta\sin\theta \quad - \left(\frac{1+\cos\theta}{2}\right)\sin^2\theta \quad + \left(\frac{1+\cos\theta}{2}\right)\cos^2\theta = 0$$

$$\sin^2\theta\left[-\cos\theta \quad -1 - \cos\theta\right] + (1+\cos\theta)\cos^2\theta = 0$$

$$\sin^2\theta = (1+\cos\theta)(1-\cos\theta)$$

$$(1-\cos\theta)(1+\cos\theta)\left[-1 - 2\cos\theta\right] + (1+\cos\theta)\cos^2\theta = 0$$

$$-(1-\cos\theta)(1+2\cos\theta) + \cos^2\theta = 0$$

$$-\left[1 + \cos\theta - 2\cos^2\theta\right] + \cos^2\theta = 0$$

$$3\cos^2\theta - \cos\theta - 1 = 0$$

$$\cos\theta = \frac{+1 \pm \sqrt{1+12}}{6} = \frac{1}{6} + \frac{\sqrt{13}}{6}$$

$$\theta^r_{max} = 39^\circ$$

4.69 Max height at $v_0 = 25 \, m/s$

$$\theta_{max} = 70.5^\circ$$

$$h_{max} = \frac{v_0^2}{2g} \left(\sin\theta_0 \cos\frac{\theta_0}{2} \right)^2 = 18.9 \, m$$

$$\text{range} = \frac{2v_0^2}{g} \cos^2\frac{\theta_0}{2} \sin\theta_0 \cos\theta_0 \qquad \theta_0 = 39^\circ$$

$$= 55 \, m$$

4.14

$$V = \frac{k}{2}\left(x^2 + 4y^2 + 9z^2\right)$$

$$F = -\nabla V = -k\left(x\,\hat{\imath} + 4y\,\hat{\jmath} + 9z\,\hat{k}\right)$$

EOM

$$m\ddot{x} = -kx \qquad \omega_{0x} = \sqrt{\frac{k}{m}} \equiv \omega_0$$

$$m\ddot{y} = -4ky \qquad \omega_{0y} = \sqrt{\frac{4k}{m}} = 2\omega_0$$

$$m\ddot{z} = -9kz \qquad \omega_{0z} = \sqrt{\frac{9k}{m}} = 3\omega_0$$

If you used $k = \pi^2 m$ $\omega_0 = \pi$, if you used $k = \pi m$ $\omega_0 = \sqrt{\pi}$

Solutions

$$x(t) = A_x\left(\cos\left(\omega_0 t + \phi_x\right)\right) = A_{x1}\cos\omega_0 t + A_{x2}\sin\omega_0 t$$

$$y(t) = A_y\cos\left(2\omega_0 t + \phi_y\right) = A_{y1}\cos 2\omega_0 t + A_{y2}\sin 2\omega_0 t$$

$$z(t) = A_z\cos\left(3\omega_0 t + \phi_z\right) = \underbrace{A_{z1}\cos 3\omega_0 t + A_{z2}\cos 2\omega_0 t}_{\text{easier}}$$

Initial Condition

$$x(0) = y(0) = z(0) = 0$$

$$\vec{v}(0) = v_0\,\hat{v}_0 = v_0\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Apply $\vec{r}(0)$ Initial Condition

$$x(0) = A_{x1} = 0$$

$$y(0) = A_{y1} = 0$$

$$z(0) = A_{z1} = 0$$

Compute velocities

$$\dot{x}(t) = A_{x2}\,\omega_0 \cos \omega_0 t$$

$$\dot{y}(t) = A_{y2}\, 2\omega_0 \cos 2\omega_0 t$$

$$\dot{z}(t) = A_{z2}\, 3\omega_0 \cos 3\omega_0 t$$

$$\dot{x}(0) = A_{x2}\,\omega_0 = \frac{v_0}{\sqrt{3}} \Rightarrow A_{x2} = \frac{v_0}{\omega_0\sqrt{3}}$$

$$\dot{y}(0) = 2\omega_0 A_{y2} = \frac{v_0}{\sqrt{3}} \Rightarrow A_{y2} = \frac{v_0}{2\omega_0\sqrt{3}}$$

$$\dot{z}(0) = 3\omega_0 A_{z2} = \frac{v_0}{\sqrt{3}} \Rightarrow A_{z2} = \frac{v_0}{3\omega_0\sqrt{3}}$$

Trajectories

$$x(t) = \frac{v_0}{\omega_0\sqrt{3}} \sin \omega_0 t$$

$$y(t) = \frac{v_0}{2\omega_0\sqrt{3}} \sin 2\omega_0 t$$

$$z(t) = \frac{v_0}{3\omega_0\sqrt{3}} \sin 3\omega_0 t$$

Periods

– The motion will repeat if an even number of periods are traversed by each component of the oscillation at the same time.

$$T_x = \frac{1}{f_x} = \frac{2\pi}{\omega_0}$$

$$T_y = \frac{1}{f_y} = \frac{2\pi}{2\omega_0} = \frac{1}{2} T_x$$

$$T_z = \frac{1}{f_z} = \frac{2\pi}{3\omega_0} = \frac{1}{3} T_x$$

So it will be $\boxed{\cancel{6}\, T_x}$ before the oscillation repeats.
at $t = T_x$, the ~~first~~ oscillator has traversed one period, the y oscillator 2 periods, and the z oscillator 3 periods.

4.17

$$\vec{E} = E_0 \hat{j} \qquad \vec{B} = B_0 \hat{k}$$

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

z

Compute $\vec{v} \times \vec{B}$

$$\vec{v} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ 0 & 0 & B_0 \end{vmatrix}$$

$$= \hat{i}\, v_y B_0 - \hat{j}\, v_x B_0$$

$$\vec{F} = \hat{i}\, q v_y B_0 + (qE_0 - v_x q B_0)\hat{j}$$

$$\boxed{\text{Note } q = -|e|}$$

Component EOM

$$m\ddot{z} = 0$$

$$m\ddot{x} = qB_0 v_y = qB_0\dot{y}$$

$$m\ddot{y} = qE_0 - qB_0\dot{x}$$

Initial Conditions

$$\vec{r}(0) = 0 \qquad \vec{v}(0) = (v_0, 0, 0)$$

Integrate

$$z(t) = 0$$

Integrate

$$\ddot{x} = \frac{qB_0}{m}\dot{y}$$

$$\int \frac{d\dot{x}}{dt}\,dt = \frac{qB_0}{m}\int \frac{dy}{dt}\,dt$$

$$\dot{x} = \frac{qB_0}{m} y + C_1$$

Apply Initial Condition

$$\dot{x}(0) = v_0 = \frac{qB_0}{m} y(0) + C_1 = C_1$$

$$\dot{x} = \frac{qB_0}{m} y + v_0$$

Integrate

$$\dot{y} = \frac{qE_0}{m} - \frac{qB_0}{m}\dot{x}$$

$$\int \frac{d\dot{y}}{dt}\,dt = \int \frac{qE_0}{m}\,dt - \int \frac{qB_0}{m}\frac{dx}{dt}\,dt$$

$$\dot{y} = \frac{qE_0}{m}t - \frac{qB_0}{m}x + C_1$$

Apply initial conditions

$$\dot{y}(0) = 0 = C_1 \qquad C_1 = 0$$

$$\dot{y} = \frac{qE_0}{m}t - \frac{qB_0}{m}x$$

Solve

$$\ddot{x} = \frac{qB_0}{m}\dot{y} = \frac{qB_0}{m}\left[\frac{qE_0}{m}t - \frac{qB_0 x}{m}\right]$$

Let $\alpha = \frac{qB_0}{m}$ $\quad \beta = \frac{qE_0}{m}$

$$\ddot{x} = \alpha\beta t - \alpha^2 x$$

$$\ddot{x} + \alpha^2 x = \alpha\beta t \qquad \omega_0 = \alpha$$

Clean that up a bit to be consistent with the notation in <u>example 4.5.1</u>.

Define $\omega^2 = \frac{q^2 B^2}{m^2}$ $\quad q^2 = e^2$

$q = -e$

$$\ddot{x} + \omega^2 x = \frac{q^2}{m^2} E_0 B_0 t$$

Write the solution as the sum of a homogeneous and particular solution $\quad x(t) = x_h(t) + x_p(t)$

$$x_h(t) = A_x \cos\omega t + B_x \sin\omega t$$

$$x_p(t) = Ct + D$$

$$\ddot{x}_p + \omega x_p = \omega^2 C = \frac{q^2}{m^2} E_0 B_0$$

$$C = \frac{\frac{q^2}{m^2} E_0 B_0}{\frac{q^2 B_0^2}{m^2}} = \frac{E_0}{B_0}$$

$$x(t) = A_x \cos\omega t + B_x \sin\omega t + \frac{E_0}{B_0}$$

$$x(0) = A_x + \frac{E_0}{B_0} = 0 \qquad A_x = -\frac{E_0}{B_0}$$

$$\ddot{x}_p + \omega^2 x_p = \omega^2 (Ct + D) = \frac{q^2}{m^2} E_0 B_0 t$$

$$\Longrightarrow \quad D = 0$$

$$C = \frac{\frac{q^2}{m^2} E_0 B_0}{\frac{q^2 B_0^2}{m}} = \frac{E_0}{B_0}$$

$$x(t) = A_x \cos \omega t + B_x \sin \omega t + \frac{E_0}{B_0} t$$

$$x(0) = 0 = A_x$$

$$\dot{x}(t) = -B_x \omega \cos \omega t + \frac{E_0}{B_0}$$

$$\dot{x}(0) = -B_x \omega + \frac{E_0}{B_0} = v_0$$

$$B_x = \frac{\frac{E_0}{B_0} - V_0}{\omega} = \frac{\frac{E_0}{B_0} - V_0}{\frac{|e| B_0}{m}}$$

$$\boxed{x(t) = \frac{1}{\omega}\left(\frac{E_0}{B_0} - v_0\right) \sin \omega t + \frac{E_0}{B_0} t}$$

$$a = \frac{1}{\omega}\left(\frac{E_0}{B_0} - v_0\right) \qquad b = \frac{E_0}{B_0}$$

$$\ddot{y} = \frac{qE_0}{m} - \frac{qB_0}{m}\dot{x}$$

but $$\dot{x} = \frac{qB_0}{m} y + v_0$$

$$\ddot{y} = \frac{qE_0}{m} - \omega^2 y - \frac{qB_0}{m} v_0$$

$$\ddot{y} + \omega^2 y = \frac{qB_0}{m}\left(\frac{E_0}{B_0} - v_0\right) = \omega\left(\frac{E_0}{B_0} - v_0\right)$$

$$y(t) = y_n(t) + y_p(t)$$

$$y_n(t) = A\cos\omega t + B\sin\omega t$$

$$y_p(t) = C$$

$$\ddot{y}_p + \omega^2 y_p = \omega^2 C = \omega\left(\frac{E_0}{B_0} - v_0\right)$$

$$C = \frac{1}{\omega}\left(\frac{E_0}{B_0} - v_0\right)$$

$$y(t) = A\cos\omega t + B\sin\omega t + \frac{1}{\omega}\left(\frac{E_0}{B_0} - v_0\right)$$

$$y(0) = 0 = A + \frac{1}{\omega}\left(\frac{E_0}{B_0} - v_0\right)$$

$$\Rightarrow A = -\frac{1}{\omega}\left(\frac{E_0}{B_0} - v_0\right)$$

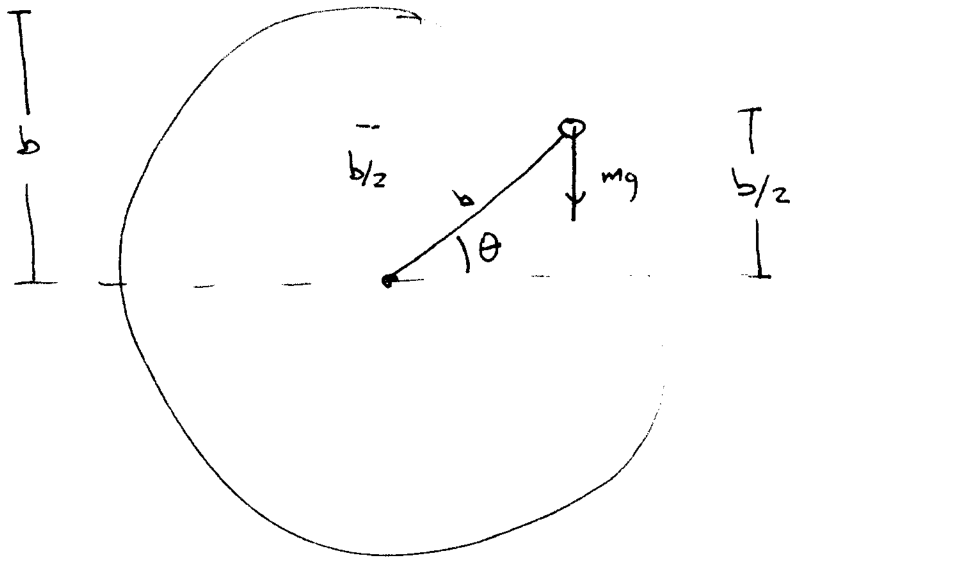
$$\dot{y}(t) = -A\omega \sin \omega t + B\omega \cos \omega t = 0$$

$$\Rightarrow B = 0$$

$$y(t) = -\frac{1}{\omega}\left(\frac{E_0}{B_0} - v_0\right)\cos \omega t + \frac{1}{\omega}\left(\frac{E_0}{B_0} - v_0\right)$$

$$= a\left(1 - \cos \omega t\right)$$

4.18



Let centerline be zero of potential energy

$$\cos \theta_0 = \frac{b/2}{b} = 1/2$$

$$\theta_0 = 60°$$

$$E_{sys} = U(\theta_0) = mg\frac{b}{2} = \frac{1}{2}mv^2 + U(\theta)$$

$$U(\theta) = mgb\sin\theta$$

The forces on the particle are mg and the normal force $\vec{R}$.

Let inward be positive

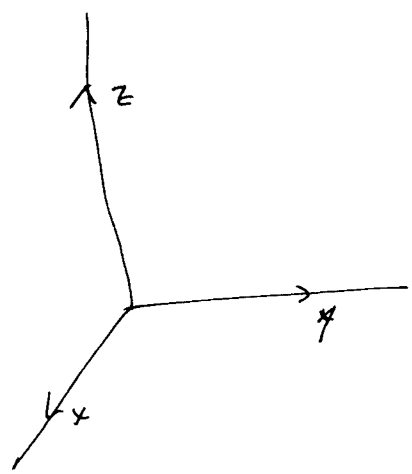
The radial forces must equal the centripetal acceleration × m

$$\frac{mv^2}{b} = -R + mg\sin\theta$$

E1

$$\vec{F} = \frac{\hat{e}_\theta}{r}$$

Compute the curl in spherical coordinates, e_r e_θ e_ϕ



$h_r = 1 \qquad h_\theta = r \qquad h_\phi\ r\sin\theta$

$$\nabla \times \vec{F} = \cancel{r}\ \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & r e_\theta & r\sin\theta\, e_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ 0 & \frac{r}{r} & 0 \end{vmatrix}$$

$$= 0$$

The particle leaves the surface when $R=0$

$$\frac{mv^2}{b} = mg\sin\theta$$

$$\frac{1}{2}mv^2 = \frac{mgb}{2}\sin\theta \qquad *$$

<u>Conserve Energy</u>

$$E_{sys} = \underbrace{T_i}_{0} + U_i = T_f + U_f$$

$$= \frac{mgb}{2} = \frac{1}{2}mv^2 + mgb\sin\theta$$

<u>Substitute *</u>

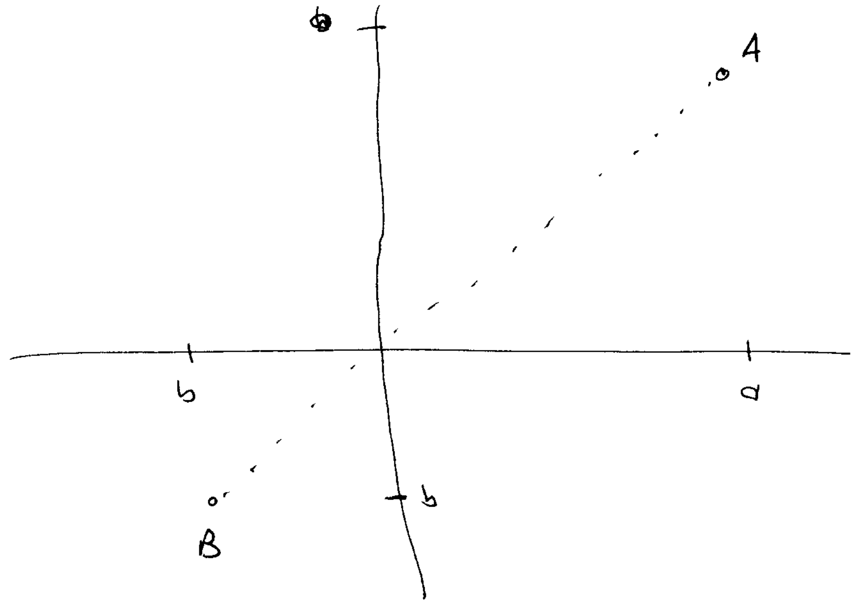
$$\frac{mgb}{2} = \frac{mgb}{2}\sin\theta + mgb\sin\theta$$

$$= \frac{3}{2}mgb\sin\theta$$

$$\sin\theta = \frac{1}{3}$$

The particle leaves the surface at height $b\sin\theta = \frac{b}{3}$

E2



$$W_{stop} = U(A) - U(B) - E_{diss}$$

$$E_{diss} = \int_{A \to B} (\mu^k mg)\, dx = \mu_k mg * (\text{Distance from } A \to B)$$

$$= \mu_k mg \left(\sqrt{2} (a+b) \right)$$

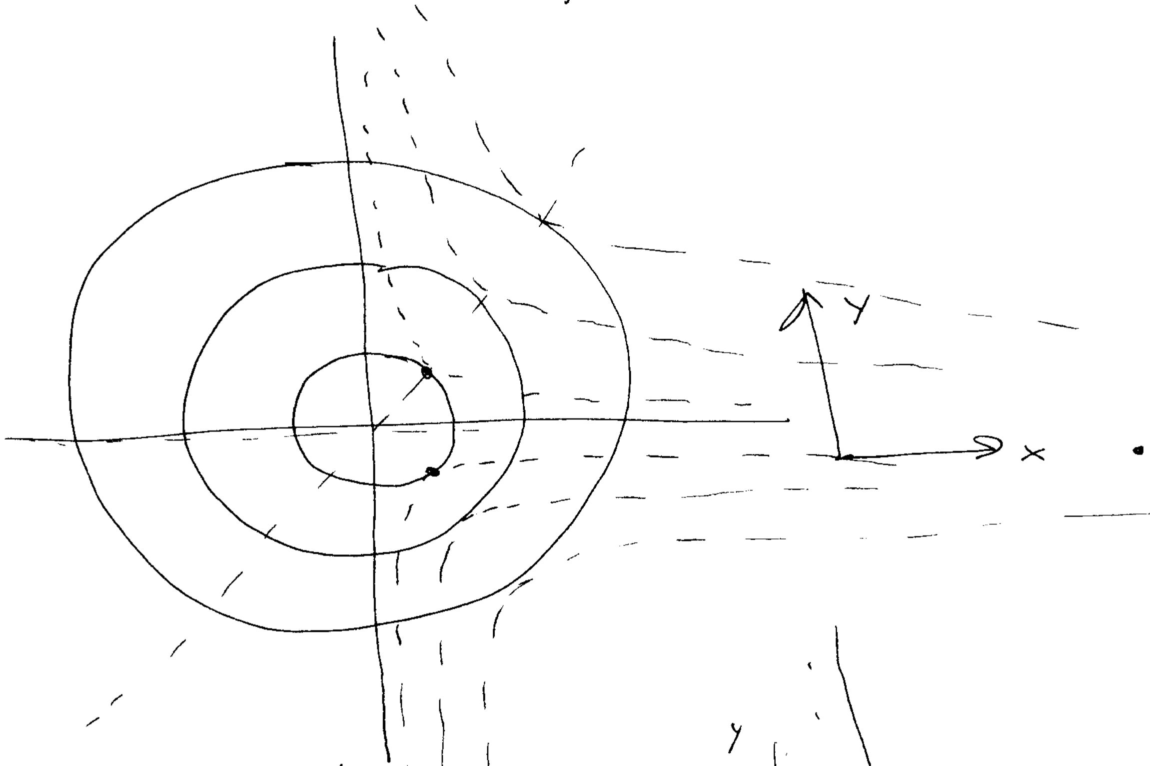
$$W_{stop} = \frac{1}{2} k (2a^2) - \frac{1}{2} k (2b^2) - \mu_k mg (\sqrt{2})(a+b)$$

$$= ka^2 - kb^2 - \mu_k mg \sqrt{2} (a+b)$$

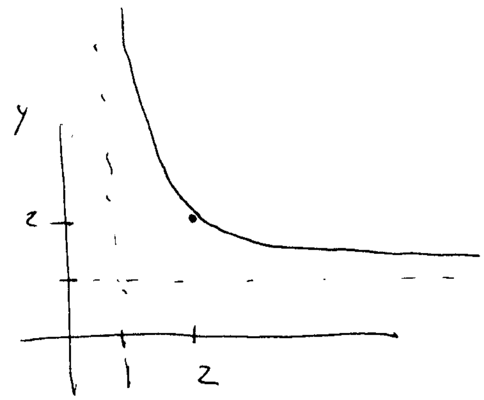
E3

——— $1/r = V_1$

- - - - - $1/x + 1/y + 1/z = V_2$



$1/r$ contours are circles



Assume $V_1 = V_2$ at $x = \emptyset, y = 1$

then $V_1 = V_2$ at

Look at assymtopes

$$\frac{1}{x} + \frac{1}{y} = 1$$

~~Ellipse~~

$$\frac{1}{y} = 1 - \frac{1}{x}$$

$$y = \frac{1}{1 - 1/x} = \frac{x}{x-1}$$