Homework 9

Due Monday 4/11/2003 at 5:00 or end of office hours

Fowles Problems

- 9.1
- 9.3
- 9.4
- 9.6
- 9.9
- 9.16
- 9.18

Bonus Problem B.1 Obtain a racket ball racket. Spin the racket about its three principle axes (they are evident by symmetry). Describe what happens.

PROBLEMS

9.1 A thin uniform rectangular plate (lamina) is of mass m and dimensions $2a$ by a. Choose a coordinate system $Oxyz$ such that the plate lies in the xy plane with origin at a corner, the long dimension being along the x-axis. Find the following:

(a) The moments and products of inertia

(b) The moment of inertia about the diagonal through the origin

(c) The angular momentum about the origin if the plate is spinning with angular rate ω about the diagonal through the origin

(d) The kinetic energy in part (c)

9.2 A rigid body consists of three thin uniform rods, each of mass m and length $2a$, held mutually perpendicular at their midpoints. Choose a coordinate system with axes along the rods.

(a) Find the angular momentum and kinetic energy of the body if it rotates with angular velocity $\boldsymbol{\omega}$ about an axis passing through the origin and the point (1, 1, 1).

(b) Show that the moment of inertia is the same for any axis passing through the origin.

(c) Show that the moment of inertia of a uniform square lamina is that given in Example 9.1.1 for any axis passing through the center of the lamina and lying in the plane of the lamina.

9.3 Find a set of principal axes for the lamina of Problem 9.1 in which the origin is

(a) At a corner

(b) At the center of the lamina

9.4 A uniform block of mass m and dimensions a by $2a$ by $3a$ spins about a long diagonal with angular velocity $\boldsymbol{\omega}$. Using a coordinate system with origin at the center of the block,

(a) Find the kinetic energy.

(b) Find the angle between the angular velocity vector and the angular momentum vector about the origin.

9.5 A thin uniform rod of length l and mass m is constrained to rotate with constant angular velocity $\boldsymbol{\omega}$ about an axis passing through the center O of the rod and making an angle α with the rod.

(a) Show that the angular momentum $\mathbf{L}$ about O is perpendicular to the rod and is of magnitude $(ml^2\omega/12)\sin\alpha$.

(b) Show that the torque vector $\mathbf{N}$ is perpendicular to the rod and to $\mathbf{L}$ and is of magnitude $(ml^2\omega^2/12)\sin\alpha\cos\alpha$.

9.6 Find the magnitude of the torque that must be exerted on the block in Problem 9.4 if the angular velocity $\boldsymbol{\omega}$ is constant in magnitude and direction.

9.7 A rigid body of arbitrary shape rotates freely under zero torque. By means of Euler's equations show that both the rotational kinetic energy and the magnitude of the angular momentum are constant, as stated in Section 9.4. (*Hint: For* $\mathbf{N} = 0$, *multiply Euler's equations (Equation 9.3.5) by* ω_1, ω_2, *and* ω_3, *respectively, and add the three equations. The result indicates the constancy of kinetic energy. Next, multiply by* $I_1\omega_1$, $I_2\omega_2$, *and* $I_3\omega_3$, *respectively, and add. The result shows that* L^2 *is constant.*)

9.8 A lamina of arbitrary shape rotates freely under zero torque. Use Euler's equations to show that the sum $\omega_1^2 + \omega_2^2$ is constant if the 1, 2 plane is the plane of the lamina. This means that the projection of $\boldsymbol{\omega}$ on the plane of the lamina is constant in magnitude, although the component ω_3 normal to the plane is not necessarily constant. (*Hint: Use the perpendicular-axis theorem.*) *What kind of lamina gives ω_3 = constant as well?*

9.9 A square plate of side a and mass m is thrown into the air so that it rotates freely under zero torque. The rotational period $2\pi/\omega$ is 1 s. If the axis of rotation makes an angle of 45° with the symmetry axis of the plate, find the period of the precession of the axis of rotation about the symmetry axis and the period of wobble of the symmetry axis about the invariable line for two cases:

(a) A thin plate

(b) A thick plate of thickness $a/4$

9.10 A rigid body having an axis of symmetry rotates freely about a fixed point under no torques. If α is the angle between the axis of symmetry and the instantaneous axis of rotation, show that the angle between the axis of rotation and the invariable line (the **L** vector) is

$$\tan^{-1}\left[\frac{(I_s - I)\tan\alpha}{I_s + I\tan^2\alpha}\right]$$

where I_s (the moment of inertia about the symmetry axis) is greater than I (the moment of inertia about an axis normal to the symmetry axis).

9.11 Since the greatest value of the ratio $I_s/I = 2$ (symmetrical lamina), show from the result of Problem 9.10 that the angle between $\boldsymbol{\omega}$ and **L** cannot exceed $\tan^{-1}(1/\sqrt{8})$ or about 19.5° and that the corresponding value of α is $\tan^{-1}\sqrt{2}$, or about 54.7°.

9.12 Find the angle between $\boldsymbol{\omega}$ and **L** for the two cases in Problem 9.9.

9.13 Find the same angle for Earth.

9.14 A space platform in the form of a thin circular disc of radius a and mass m (flying saucer) is initially rotating steadily with angular velocity $\boldsymbol{\omega}$ about its symmetry axis. A meteorite strikes the platform at the edge, imparting an impulse $\boldsymbol{P}$ to the platform. The direction of $\boldsymbol{P}$ is parallel to the axis of the platform, and the magnitude of $\boldsymbol{P}$ is equal to $ma\omega/4$. Find the resulting values of the precessional rate Ω, the wobble rate $\dot{\phi}$, and the angle α between the symmetry axis and the new axis of rotation.

9.15 A Frisbee is thrown into the air in such a way that it has a definite wobble. If air friction exerts a frictional torque $-c\boldsymbol{\omega}$ on the rotation of the Frisbee, show that the component of $\boldsymbol{\omega}$ in the direction of the symmetry axis decreases exponentially with time. Show also that the angle α between the symmetry axis and the angular velocity vector $\boldsymbol{\omega}$ decreases with time if I_s is larger than I, which is the case for a flat-type object. Thus, the degree of wobble steadily diminishes if there is air friction.

9.16 A simple gyroscope consists of a heavy circular disc of mass m and radius a mounted at the center of a thin rod of mass $m/2$ and length a. If the gyroscope is set spinning at a given rate S, and with the axis at an angle of 45° with the vertical, there are two possible values of the precession rate $\dot{\phi}$ such that the gyroscope precesses steadily at a constant value of $\theta = 45°$.

(a) Find the two numerical values of $\dot{\phi}$ when S = 900 rpm and a = 10 cm.

(b) How fast must the gyroscope spin in order to sleep in the vertical position? Express the results in revolutions per minute.

9.17 A pencil is set spinning in an upright position. How fast must the spin be for the pencil to remain in the upright position? Assume that the pencil is a uniform cylinder of length a and diameter b. Find the value of the spin in revolutions per second for $a = 20$ cm and $b = 1$ cm.

9.18 A bicycle wheel of diameter 30 in. rolls along the ground. How fast must it roll to remain upright? Assume that half the mass of the wheel is on the periphery (rim), one fourth of the mass is in the spokes, and the remainder is concentrated at the center (hub). Compare the result with that obtained if the spokes and hub are ignored.

9.19 A rigid body rotates freely under zero torque. By differentiating the first of Euler's equations with respect to t, and eliminating $\dot{\omega}_2$ and $\dot{\omega}_3$ by means of the second and third of Euler's equations, show that the following result is obtained:

$$\ddot{\omega}_1 + K_1\omega_1 = 0$$

in which the function K_1 is given by

$$K_1 = -\omega_2^2\left[\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}\right] + \omega_3^2\left[\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\right]$$

Two similar pairs of equations are obtained by cyclic permutations: $1 \rightarrow 2$, $2 \rightarrow 3$, $3 \rightarrow 1$. In the preceding expression for K_1 both quantities in brackets are *positive constants* if $I_1 < I_2 < I_3$, or if $I_1 > I_2 > I_3$. Discuss the question of the growth of ω_1 (stability) if initially ω_1 is very small and (a) $\omega_2 = 0$ and ω_3 is large: initial rotation is very nearly about the 3-axis, and (b) $\omega_3 = 0$ and ω_2 is large: initial rotation is nearly about the 2-axis. (Note: This is an analytical method of deducing the stability criteria illustrated in Figure 9.[illegible])

9.20 A rigid body consists of six particles, each of mass m, fixed to the ends of three light rods of length $2a$, $2b$, and $2c$, respectively, the rods being held mutually perpendicular to one another at their midpoints.

(a) Show that a set of coordinate axes defined by the rods are principal axes, and write down the inertia tensor for the system in these axes.

(b) Use matrix algebra to find the angular momentum and the kinetic energy of the system when it is rotating with angular velocity $\boldsymbol{\omega}$ about an axis passing through the origin and the point (a, b, c).

9.21 Work Problems 9.1 and 9.4 using matrix methods.

9.22 A uniform rectangular block of dimensions $2a$ by $2b$ by $2c$ and mass m spins about a long diagonal. Find the inertia tensor for a coordinate system with origin at the center of the block and with axes normal to the faces. Find also the angular momentum and the kinetic energy. Find also the inertia tensor for axes with origin at one corner.

9.23 Show that the z-component of angular momentum L_z of the simple gyroscope discussed in Section 9.7 is given by Equation 9.7.7.

9.24 If the top discussed in Section 9.7 is set spinning very rapidly ($\dot{\psi} \gg 0$) its rate of precession will slow ($\dot{\phi} \approx 0$) and the angular difference $\theta_2 - \theta_1$ between the limits of its nutational motion will be small. Assuming this condition, show that the top can be made to precess without nutation if its motion is started with $\dot{\theta}|_{\theta=\theta_1} = 0$ and $\dot{\phi}|_{\theta=\theta_1} = mgl/L_z$, where $L_z = I_3 S$.

9.25 Formaldehyde molecules (CH_2O) have been detected in outer space by the radiation they emit when they change rotational states. Assume that the molecule is a rigid body shaped like a regular tetrahedron whose faces make equilateral triangles. The masses of the oxygen, carbon, and hydrogen atoms are 16, 12, and 1 AMU, respectively

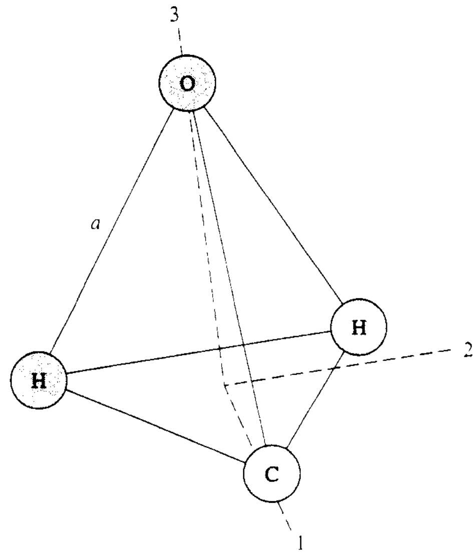


Figure P9.25

(a) Show that a coordinate system whose: 3-axis passes through the oxygen atom and its projection onto the face formed by the carbon and two hydrogen atoms; 1-axis passes through that point and the carbon atom; 2-axis is parallel to the line connecting the two hydrogen atoms are principal axes of the molecule, as shown in Figure P9.25.

(b) Write down the inertia tensor for the molecule about these axes.

(c) Assume that the molecule rotates with angular velocity given by

$$\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_3 \mathbf{e}_3$$

Show that motion is stable if the molecule rotates mostly about either the 3-axis or the 2-axis (i.e., if ω_1 and ω_2 are small compared with ω_3 *or* if ω_1 and ω_3 are small compared with ω_2) but not stable if it rotates mostly about the 1-axis.

COMPUTER PROBLEMS

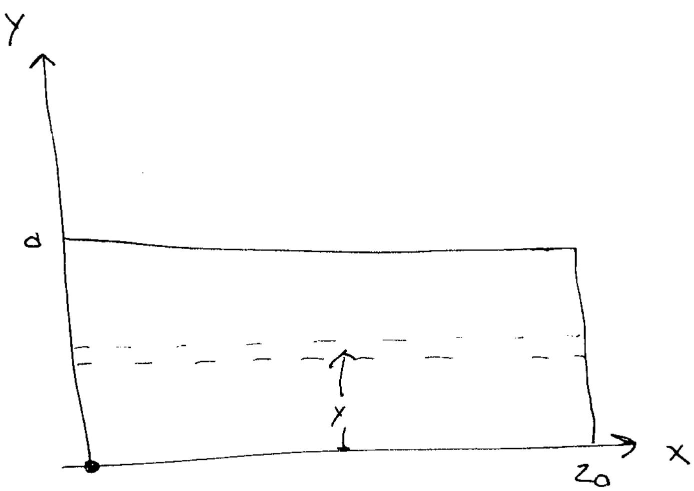
C 9.1 Consider the spinning top discussed in Examples 9.7.1 and 9.7.2. Suppose that it is set spinning at 35 rev/s and initially its spin axis is held fixed at an angle $\theta_0 = 60°$. The axis is then released and the top starts to topple over. As it falls, its axis starts to precess as well as to nutate between two limiting polar angles θ_1 and θ_2.

(a) Calculate the two limits θ_1 and θ_2.

(b) Estimate the period of nutation analytically. (*Hint: Make approximations, where necessary, in the expression given in the text for $\dot{u}$ and then integrate.*)

(c) Estimate the average period of precession analytically. (*Hint: Make approximations in the expression for $\dot{\phi}$ given in the text.*)

(9.1)



Moment of Inertia about x- axis, I_{xx}

$$I_{xx} = \int y^2 dm = \int_0^a y^2 (dy\, \sigma\, 2a) \qquad m = 2\sigma a^2$$

$\sigma \equiv$ mass density

$$= 2\sigma a \int_0^a y^2 dy = \frac{2\sigma a}{3} y^3 \Big|_0^a = \frac{2\sigma a^4}{3} = \frac{ma^2}{3}$$

$$I_{yy} = \sigma \int_0^{2a} dx\, x^2 (a) = \frac{\sigma a x^3}{3} \Big|_0^{2a} = \frac{8\sigma a^4}{3} = \frac{4a^2 m}{3}$$

Perpendicular Axis Thrm (Can apply because lamina) -

$$I_{zz} = I_{xx} + I_{yy}$$

$$= \frac{ma^2}{3} + \frac{4a^2 m}{3} = \frac{5ma^2}{3}$$

<u>Products of Inertia</u> - I_{yz}, $I_{xz} = 0$ because $z = 0$.

$$I_{xy} = -\int xy\,dm \qquad \text{Now let } dm = \sigma\,dx\,dy$$

$$= -\sigma \int_0^{2a}\int_0^{a} xy\,dx\,dy = -\sigma \int_0^{2a} x\,dx\ \frac{y^2}{2}\Big|_0^{2a}$$

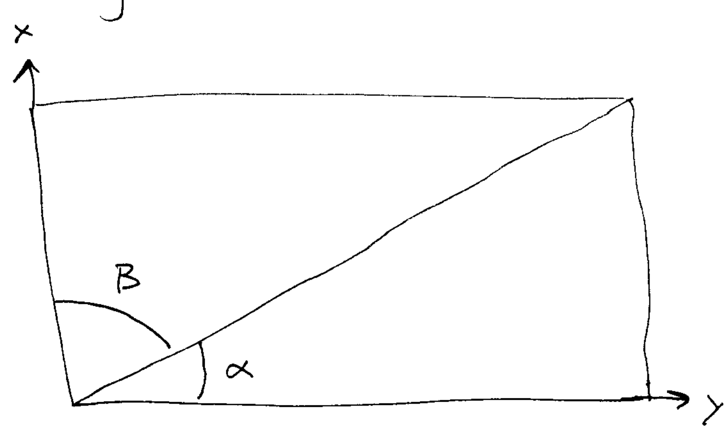
$$= \frac{-\cancel{4}\sigma a^2}{2}\int_0^{2a} x\,dx = -\frac{\cancel{2}\sigma a^2}{2}\,\frac{x^2}{2}\Big|_0^{2a} = -\sigma a^4 = \frac{-2\sigma a^4}{2}$$

$$= \frac{-ma^2}{2}$$

(b)

$$\underline{I} = \begin{pmatrix} \frac{ma^2}{3} & -\frac{ma^2}{2} & 0 \\ -\frac{ma^2}{2} & \frac{4ma^2}{3} & 0 \\ 0 & 0 & \frac{5ma^2}{3} \end{pmatrix}$$

The unit vector for a diagonal through the origin



$$\hat{n} = (\cos\alpha, \cos B, 0)$$

$$\cos\alpha = \frac{2a}{\sqrt{a^2+(2a)^2}} = \frac{2}{\sqrt{5}}$$

$$\cos\beta = \frac{a}{\sqrt{a^2+(2a)^2}} = \frac{1}{\sqrt{5}}$$

$$\vec{n} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$

$$I_{\text{diagonal}} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{pmatrix} \begin{pmatrix} \frac{ma^2}{3} & -\frac{ma^2}{2} & 0 \\ -\frac{ma^2}{2} & \frac{4ma^2}{3} & 0 \\ 0 & 0 & \frac{5ma^2}{3} \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}$$

Also used for angular momentum

$$= \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \end{pmatrix} \begin{pmatrix} \frac{2ma^2}{3\sqrt{5}} - \frac{ma^2}{2\sqrt{5}} + 0 \\ -\frac{ma^2}{\sqrt{5}} + \frac{4ma^2}{3\sqrt{5}} + 0 \\ 0 \end{pmatrix}$$

$$I_{\text{diagonal}} = \frac{4ma^2}{15} - \frac{ma^2}{5} - \frac{ma^2}{5} + \frac{4ma^2}{15}$$

$$= ma^2\left(\frac{8}{15} - \frac{2}{5}\right) = ma^2\left(\frac{8}{15} - \frac{6}{15}\right)$$

$$= \frac{2}{15}ma^2$$

(c) $$\vec{L} = \overleftrightarrow{I}\,\vec{\omega} = \omega\left(\overleftrightarrow{I}\,\vec{n}\right)$$

$$\vec{\omega} = \omega\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)$$

Use first piece of multiplication from part (b)

$$\vec{L} = ma^2\omega\left[\frac{2}{3\sqrt{5}} - \frac{1}{2\sqrt{5}}, \frac{4}{3\sqrt{5}} - \frac{1}{\sqrt{5}}, 0\right]$$

$$= \frac{ma^2\omega}{\sqrt{5}}\left[\frac{1}{6}, \frac{1}{3}, 0\right]$$

(d) $$T = \frac{1}{2} I_{diag}\, \omega^2$$

$$= \frac{1}{2}\left(\frac{2}{15} m a^2\right)\omega^2$$

$$= \frac{1}{15} m a^2 \omega^2$$

You could also obtain this through

$$T = \frac{1}{2}\, \vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega}$$

$$= \frac{1}{2}\omega^2\, \vec{n} \cdot \overleftrightarrow{I} \cdot \vec{n}$$

9.3

$$I = \frac{ma^2}{6} \begin{pmatrix} 2 & -3 & 0 \\ -3 & 8 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

We need to diagonalize the matrix. It is already diagonal about the z-axis. So diagonalize

$$\begin{pmatrix} 2 & -3 \\ -3 & 8 \end{pmatrix} = I'$$

$$\overleftrightarrow{I}' \vec{x}_1 = \lambda'_1 \vec{x}_1$$

Let $\lambda' = \frac{ma^2}{6}\lambda$, so $ma^2/6$ cancels out.

If $\vec{x}_1$ is an eigenvector (a principle axis) and λ_1 its eigenvalue

$$\det \begin{pmatrix} 2-\lambda & -3 \\ -3 & 8-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(8-\lambda) - 9 = 0$$

$$16 - 10\lambda + \lambda^2 - 9 = 0$$

$$\lambda^2 - 10\lambda + 7 = 0$$

$$\cancel{(\lambda - 3)(\lambda - 7)}$$

$$\lambda = \frac{10 \pm \sqrt{100 - 28}}{2}$$

$$= \frac{10 \pm \sqrt{72}}{2}$$

$$= 5 \pm \sqrt{18}$$

$$\lambda_1 = 9.243$$

$$\lambda_2 = 0.7573$$

For λ_1

$$\begin{pmatrix} 2-\lambda_1 & -3 \\ -3 & 8-\lambda_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

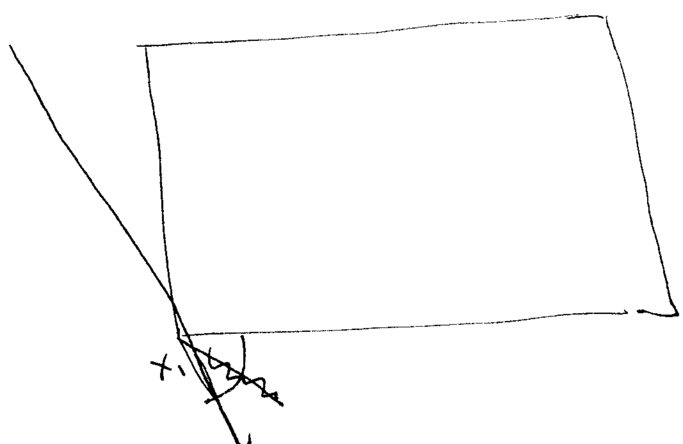
$$\begin{pmatrix} -7.243 & -3 \\ -3 & -1.243 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$-7.243x - 3y = 0$$

$$-3x - 1.243y = 0$$

Let $x = 1$, $\quad y = -\dfrac{3}{1.243} = -2.41$

$$\hat{x}_1 = \frac{(1, -2.41, 0)}{2.61} = (0.383, -.922, 0)$$



For λ_2

$$\begin{pmatrix} 1.2427 & -3 \\ -3 & 7.2427 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$1.2427x - 3y = 0$$

$$-3x + 7.2427y = 0$$

Let $x = 1$, $y = \dfrac{1.2427}{3} = 0.414$

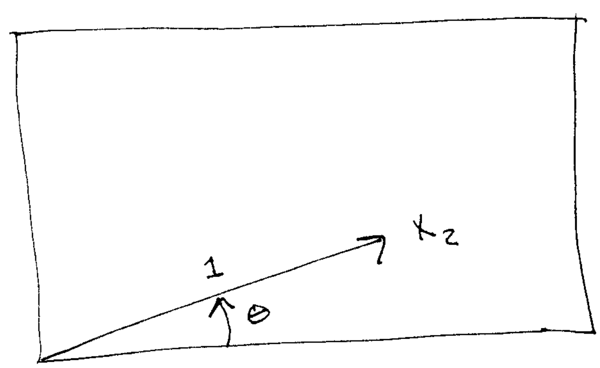
$$\vec{x}_2 = (1, 0.414, 0)$$

$$= (0.924, 0.383, 0)$$

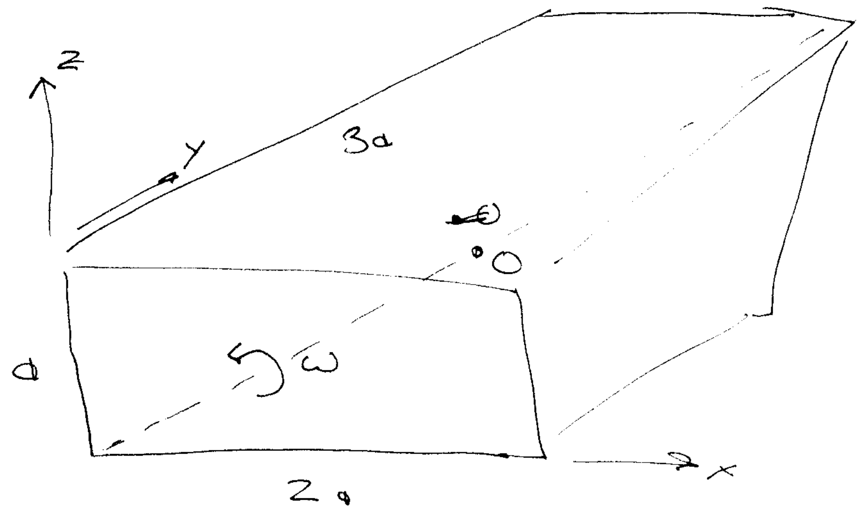
$$\hat{x}_2 = \cancel{(0.8535, 0.353, 0)}$$

$$\cos\alpha = 0.924$$

$$\alpha = 22.5^\circ$$



9.21



If the block is uniform, the moment of inertia about an axis ⊥ to a face of the block is the same as a lamina.

$$I_{xx} = \frac{a^2 + (3a)^2}{12} \qquad (a \times 3a)$$

$$= \frac{10a^2}{12} = \frac{5}{6}a^2$$

(The moment of inertia of a rectangular lamina is $\dfrac{a^2+b^2}{12}$)

$$I_{yy} = \frac{a^2 + (2a)^2}{12} \qquad (a \times 2a)$$

$$= \frac{5a^2}{12}$$

$$I_{zz} = \frac{(2a)^2 + (3a)^2}{12} \qquad (2a \times 3a)$$

$$= \frac{13\,a^2}{12}$$

Those three axes are the moment of inertia by symmetry.

$$I = \begin{pmatrix} \frac{5}{6}a^2 & 0 & 0 \\ 0 & \frac{5}{12}a^2 & 0 \\ 0 & 0 & \cancel{\frac{8}{6}}\ \frac{13}{12}a^2 \end{pmatrix}$$

$$I = \frac{1}{12}a^2 \begin{pmatrix} 10 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 13 \end{pmatrix}$$

Now build axis of rotation, unit vector from origin to far corner is

$$\vec{n} = (2a,\ 3a,\ a)$$

$$|\vec{n}| = \sqrt{4+9+1}\ a = \sqrt{14}\ a$$

$$\hat{n} = \left(\frac{2}{\sqrt{14}} , \frac{3}{\sqrt{14}} , \frac{1}{\sqrt{14}} \right)$$

$$\vec{\omega} = \omega \hat{n}$$

$$T = \frac{1}{2} \vec{\omega} \cdot \vec{L}$$

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$$

$$= \frac{1}{12} a^2 \left(\frac{20}{\sqrt{14}} , \frac{15}{\sqrt{14}} , \frac{13}{\sqrt{14}} \right) \omega$$

$$\vec{\omega} \cdot \vec{L} = \frac{1}{12} a^2 \omega^2 \left[\frac{40 + 45 + 13}{14} \right]$$

$$= \frac{1}{12} a^2 \omega^2 \left[\frac{98}{14} \right]$$

$$= \frac{7}{12} a^2 \omega^2$$

$$T = \frac{1}{2} \vec{\omega} \cdot \vec{L} = \frac{7}{24} a^2 \omega^2$$

$$98 = 2 \cdot 7 \cdot 7$$
$$= 7 \cdot 14$$

(b) $$|\vec{\omega}| = \omega\left(\frac{4+9+1}{14}\right) = \omega$$

Doh!

$$|\vec{L}| = \frac{1}{12}a^2\left(\frac{400+225+169}{14}\right)^{1/2}$$

$$= \frac{1}{12}a^2\omega\left(\frac{397}{7}\right)^{1/2}$$

$$\cos\theta = \frac{\vec{\omega}\cdot\vec{L}}{|\omega||\vec{L}|} = \frac{2T}{|\omega||\vec{L}|}$$

$$= \frac{\frac{7}{12}a^2\omega^2}{\omega\left(\frac{1}{12}\right)a^2\omega\left(\frac{397}{7}\right)^{1/2}}$$

$$= \frac{7}{\left(\frac{397}{7}\right)^{1/2}} = 0.9295$$

$$\theta = 21.64^\circ$$

9.6 Euler's Eqns

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} I_1 \dot{\omega}_1 \\ I_2 \dot{\omega}_2 \\ I_3 \dot{\omega}_3 \end{pmatrix} + \begin{pmatrix} \omega_2 \omega_3 (I_3 - I_2) \\ \omega_3 \omega_1 (I_1 - I_3) \\ \omega_1 \omega_2 (I_2 - I_1) \end{pmatrix}$$

$\vec{\omega}$ constant $\dot{\vec{\omega}} = 0$

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \frac{1}{12} a^2 \begin{pmatrix} \omega_2 \omega_3 (13 - 5) \\ \omega_3 \omega_1 (10 - 13) \\ \omega_1 \omega_2 (5 - 10) \end{pmatrix}$$

In principle axis basis

$$\vec{N}^P = \frac{1}{12} a^2 \left(8\omega_2\omega_3 , -3\omega_3\omega_1 , -5\omega_1\omega_2 \right)$$

$$|\vec{N}| = \frac{1}{12} a^2 \sqrt{64\omega_2^2 \omega_3^2 + 9\omega_3^2\omega_1^2 + 25\omega_1^2\omega_2^2}$$

same in all coordinate systems.

From above, $\qquad \vec{\omega} = \dfrac{\omega}{\sqrt{14}} \left(2, 3, 1 \right)$

$$|\vec{W}| = \frac{a^2 \omega^2}{(12)(14)} \left(64(3)^2(1)^2 + 9(1)^2(2)^2 + 25(2)^2(3)^2 \right)^{1/2}$$

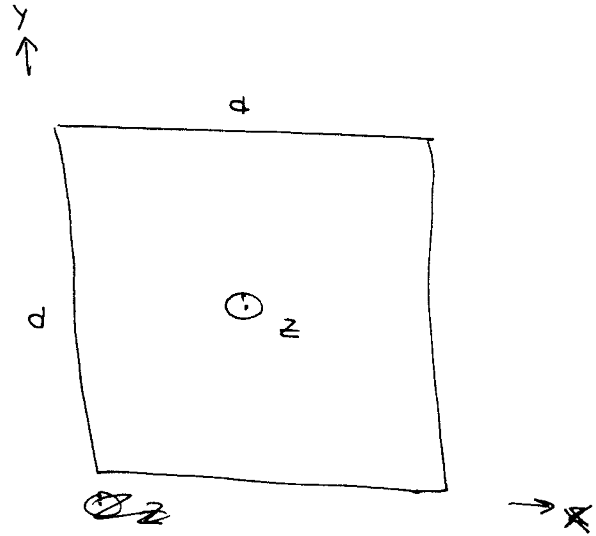
$$= \frac{a^2 \omega^2}{(12)(14)} \sqrt{1512} = \frac{a^2 \omega^2}{(12)(14)} 6\sqrt{42}$$

$$= \frac{a^2 \omega^2 \sqrt{42}}{28}$$

9.9

$$T = \frac{2\pi}{\omega} = 1\,s$$

$$\omega = 2\pi\, s^{-1}$$



Principle Axes are evident by symmetry.

$$\overleftrightarrow{I} = M \begin{pmatrix} a^2/12 & 0 & 0 \\ 0 & a^2/12 & 0 \\ 0 & 0 & \frac{a^2}{6} \end{pmatrix}$$

From table, moment of rectangular lamina $I = \frac{a^2+b^2}{12}$

From perpendicular axis thm, $I_z = I_x + I_y$

The body has an axis of symmetry $I_s = \frac{Ma^2}{6}$

Moment about axes normal to symmetry axes, $I = I_1 = I_2 = \frac{Ma^2}{12}$

For a body with a symmetry axis, from 9.6.12,

$$\dot{\phi} = \omega\left[1 + \left(\frac{I_s^2}{I^2} - 1\right)\cos^2\alpha\right]^{1/2}$$

$$= (2\pi s^{-1})\left[1 + \left(\frac{\left(\frac{Ma^2}{6}\right)^2}{\left(\frac{Ma^2}{12}\right)^2} - 1\right)\cos^2 45\right]^{1/2}$$

$$= (2\pi s^{-1})\left[1 + \frac{3}{2}\right]^{1/2}$$

$$= 2\sqrt{\frac{5}{2}}\,\pi s^{-1}$$

Period of Wobble = $T = \frac{2\pi}{\dot{\phi}} = \sqrt{\frac{2}{5}}$ s

$= 0.63$ s

(b) Rate of Precession ω about I_s

$T = \frac{2\pi}{\Omega} = \sqrt{2}$ s $= 1.414$ s

From eqn 9.5-8, the precession rate of the angular velocity about the symmetry axis (the body cone)

$$\Omega = \left(\frac{I_s}{I} - 1\right) \omega \cos \alpha$$

where α is the angle the axis of rotation makes with the symmetry axis. Here $\alpha = 45^\circ$

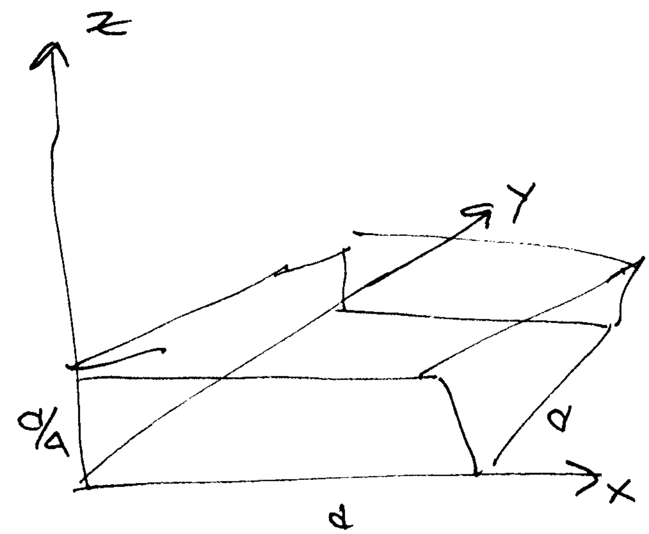
$$\Omega = \left(\frac{\frac{Ma^2}{6}}{\frac{Ma^2}{12}} - 1\right)\left(2\pi s^{-1}\right) \cos 45$$

$$= (2 - 1)(2\pi s^{-1}) \frac{1}{\sqrt{2}}$$

$$= \frac{\pi}{\sqrt{2}} s^{-1} = \frac{2\pi}{\sqrt{8}} s^{-1}$$

Now we need the wobble rate, the precession of the body axes about the "invariable line", the direction of the ~~moment of inertia~~ angular momentum.

(b) Thick plate $a \times a \times a/4$



$$I_s = I_{zz} = \frac{Ma^2}{6} \qquad I = I_{xx} = M\left(\frac{a^2 + (a/4)^2}{12}\right)$$

$$= \frac{17\, a^2 M}{12 \cdot 16} = I_{yy}$$

Precession rate $\vec{\omega}$ about 3,

$$\Omega = \left(\frac{I_s}{I} - 1\right) \omega \cos 45^\circ$$

$$= \frac{1}{\sqrt{2}}\left(\frac{\frac{Ma^2}{6}}{\frac{17Ma^2}{12 \cdot 16}} - 1\right)\omega$$

$$\Omega = \frac{1}{\sqrt{2}}\left(\frac{32}{17} - 1\right)\omega = \frac{1}{\sqrt{2}}\,\frac{15}{17}\,\omega$$

$$T = \frac{2\pi}{\Omega} = \frac{17}{15\sqrt{2}}\ s$$

$$= 0.80\ s$$

The rate of wobble is

$$\dot{\phi} = \omega\left(1 + \left(\left(\frac{I_s}{I}\right)^2 - 1\right)\cos^2\alpha\right)^{1/2}$$

$$= \omega\left(1 + \frac{1}{2}\left(\left(\frac{\frac{Ma^2}{6}}{\frac{17a^2M}{16\cdot 12}}\right)^2 - 1\right)\right)^{1/2}$$

$$= \omega\left(1 + \frac{1}{2}\left(\left(\frac{32}{17}\right)^2 - 1\right)\right)^{1/2}$$

$$= \omega\left(1 + \cancel{4}\frac{1}{2(17)^2}\left(32^2 - 17^2\right)\right)^{1/2}$$

$$= \omega\left(1 + \frac{735}{2(17)^2}\right)^{1/2} = 1.51\,\omega$$