Mechanics Spring 2003 - Test 3

Problem 3.1 A pipe (a thin hollow right cylinder) of mass M and radius $a = 5$cm is placed on an incline with coefficient of static friction $\mu_s = \tan 60°$ and coefficient of dynamic friction $\mu_k = \mu_s/4$. The angle of the incline is $45°$. This problem is to be worked from the basic equations of motion not from any derived formula in the text.

(a) Does the pipe slip or roll without slipping immediately after it is released? Justify.

The horizontal distance from the starting location to the bottom of the ramp is $d = 0.5$m.

(b) Compute the time it takes the pipe to roll (or slip) down the ramp.

(c) Compute the torque about the center of mass on the pipe as it reaches the bottom of the ramp.

Problem 3.2 Consider a rigid body made out of tinker toys (a model of the object is at the front of the class). The body consists of five masses, m, at locations $\vec{r}_1 = (0,0,0)$, $\vec{r}_2 = (a,0,0)$, $\vec{r}_3 = (0,a,0)$, $\vec{r}_4 = (a,a,0)$, and $\vec{r}_5 = (0,0,2a)$.

(a) Compute the inertia tensor about the origin.

(b) Compute the center of mass.

Suppose the body rotates about a line from the origin to mass 4 at an angular velocity of ω.

(c) Calculate the angular momentum.

(d) Calculate the kinetic energy.

(e) Calculate the angle between the axis of rotation and the angular momentum.

Problem 3.3 A water rocket (Soda bottle filled with water and pumped with a bicycle pump) ejects mass at a constant rate (under the assumption of constant pressure),

$$\frac{dm}{dt} = -\gamma = -\rho A|v_{rel}|$$

where $\rho = 1000\text{kg}/m^3$ is the density of water, $A = 3.14 \times 10^{-4}\text{m}^2$ is the area of the nozzle (bottleneck), and v_{rel} is the relative velocity between the rocket and the ejected water. The relative velocity can be found by assuming there are no energy losses as the water is ejected,

$$v_{rel} = -\sqrt{\frac{2P}{\rho}},$$

where P is the pressure in the rocket. Therefore the thrust,

$$\frac{dm}{dt}v_{rel} = 2PA$$

is constant. The rocket is initially filled with $M_w = 2$kg of water and has an empty mass of $M_r = 0.1$kg. The total initial mass is $M = M_r + M_w$.

(a) How much pressure is required for liftoff? You may leave it in Pascals, but if you want a more familiar unit $1.013 \times 10^5\text{Pa} = 14.7\text{lb/in}^2$.

For the remaining parts use $P = 100\text{lb/in}^2 = 7 \times 10^5\text{Pa}$, the pressure my bike pump can provide.

(b) Find the time t_e to eject all the water.

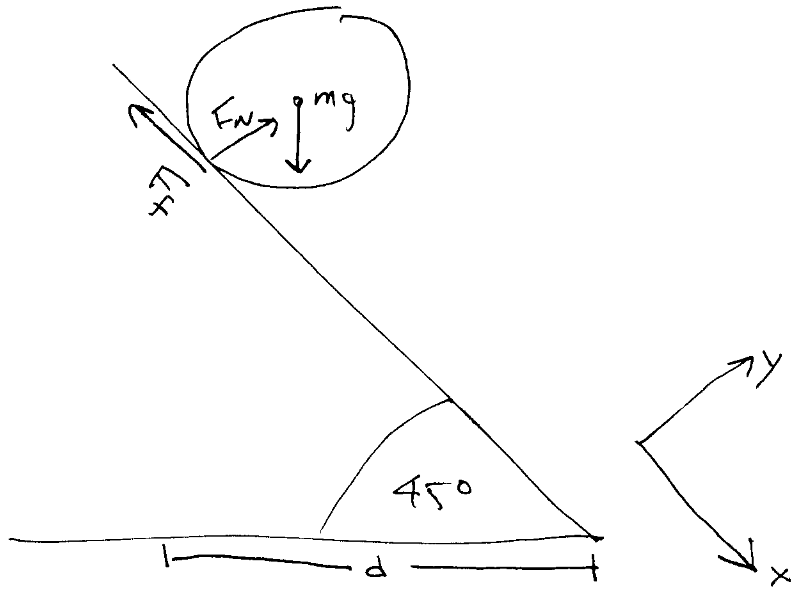
(c) Find the velocity as a function of time.

(d) Numerically and symbolically calculate the velocity at the time all the water is ejected. This should be the maximum velocity.

(e) Set up the integral to calculate the height as a function of time including appropriate limits.

Bonus Problem What the middle name of any of the mechanics students who were in Bernadette's hospital room Monday night?

3.1



EOM

$$\vec{F} = m\vec{a}_{cm}$$

$$\vec{N}' = \frac{dL'}{dt} = I\dot{\omega} = ma^2\dot{\omega}$$

The moment of inertia of a cylinder is $I = ma^2$

x-component

$$m\ddot{x} = -F_f + mg\sin 45^\circ$$

y-component

$$m\ddot{y} = 0 = F_N - mg\cos\theta$$

$$F_N = mg\cos\theta = \frac{mg}{\sqrt{2}}$$

Suppose cylinder doesn't slip, $F_f < \mu_s mg\cos\theta$

$$= mg\frac{\tan 60^\circ}{\sqrt{2}}$$

The torque on the cylinder $N' = aF_f = ma^2\dot{\omega}$

$$F_f = ma\dot{\omega}$$

If it doesn't slip, $\frac{x_{cm}}{a} = \theta$

$$\frac{v_{cm}}{a} = \omega$$

$$\frac{a_{cm}}{a} = \dot{\omega}$$

$$F_f = m a_{cm}$$

$$m a_{cm} = -m a_{cm} + \frac{mg}{\sqrt{2}}$$

$$a_{cm} = \frac{g}{2\sqrt{2}}$$

$$F_f = \frac{mg}{2\sqrt{2}} \overset{?}{<} mg\,\frac{\tan 60}{\sqrt{2}}$$

$$0.35mg < mg(1.22)$$

$\boxed{\text{So object rolls without slipping.}}$

(b) We have already computed $\ddot{x} = g/2\sqrt{2}$.

If the horizontal distance to be traversed is $d = 0.5m$,

the the distance in the x direction is

$$d_x = d/\cos 45 = \; = \sqrt{2}\,d$$

For motion under constant acceleration,

$$d_x = \frac{1}{2}\ddot{x}\,t^2$$

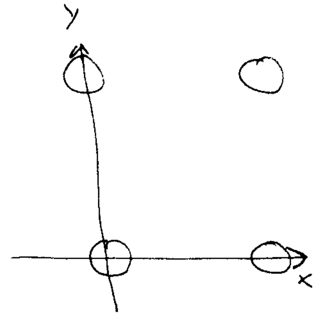
$$t = \sqrt{\frac{2d_x}{\ddot{x}}} = \sqrt{\frac{2\sqrt{2}\;0.5m}{g/2\sqrt{2}}}$$

$$= \sqrt{\frac{4m}{g}} = 0.639s.$$

(c) The angular acceleration $\dot{\omega} = \frac{a_{cm}}{a}$, is constant, $N' = I\dot{\omega} = m\dot{a}^2 \frac{a_{cm}}{a}$

$$= \frac{mag}{2\sqrt{2}}$$

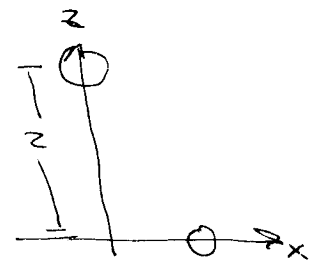
(3.2) $I_{xx} = \sum_i m_i (y_i^2 + z_i^2)$



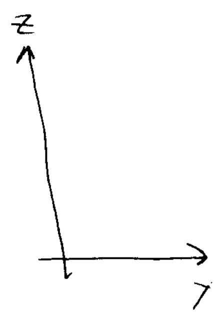
$= m_3(y_3^2 + z_3^2) + m_4(y_4^2 + \cancel{z_4}^2)$

$+ m_5(y_5^2 + \cancel{z_5}^2)$

$= ma^2 + \cancel{4}a^2m + 4a^2m$



$I_{xx} = 6a^2m$



$I_{yy} = 6a^2m$ (by symmetry

$I_{zz} = \sum m_i (x_i^2 + y_i^2)$

$= m_2(x_2^2 + y_2^2) + m_3(x_3^2 + y_3^2) + m_4(x_4^2 + y_4^2)$

$= ma^2 + ma^2 + 2a^2m = 4a^2m$

Products of inertia exist when two components of r are non-zero.

$I_{yz} = I_{zy} = I_{xz} = I_{zx} = 0$

$$I_{xy} = -\sum m_i x_i y_i$$

$$= -m_4 x_4 y_4 = -a^2$$

$$I = m\begin{pmatrix} 6a^2 & -a^2 & 0 \\ -a^2 & 6a^2 & 0 \\ 0 & 0 & 4a^2 \end{pmatrix} = ma^2\begin{pmatrix} 6 & -1 & 0 \\ -1 & 6 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

(b) <u>Center of mass</u>

$$\vec{r}_{cm} = \frac{1}{M}\sum m_i \vec{r}_i = \frac{m}{5m}\sum \vec{r}_i$$

$$= \frac{1}{5}(2a, 2a, 2a) = \frac{2a}{5}(1, 1, 1)$$

(c) Axis of rotation, $\vec{\omega} = \omega\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$$\vec{L} = \overleftrightarrow{I}\,\vec{\omega}$$

$$= ma^2\omega \begin{pmatrix} 6 & -1 & 0 \\ -1 & 6 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$= \frac{ma^2\omega}{\sqrt{2}}\left(5, 5, 0\right)$$

(d) $T = \frac{1}{2}\,\vec{\omega}\cdot\vec{L}$

$$= \frac{1}{2}\left[\left(\frac{ma^2\omega}{\sqrt{2}}\right)(5, 5, 0)\right]\cdot\left[\omega\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\right]$$

$$= \frac{5}{4}ma^2\omega^2$$

(e) $\vec{L} \parallel \vec{\omega}$ since $(5, 5, 0)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ are in the same direction.

(f) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ is the direction of one of the principle axes of inertia.

3.3 Water rocket, if V_{rel} is the exhaust speed, $\dot{m} = -\rho A V_{rel}$.

If there are no losses (and pressure is constant) the work done on the water

$$W = P\delta V = \frac{1}{2}(\rho \delta V) V_{rel}^2$$

$$V_{rel}^2 = \frac{2P}{\rho} \qquad V_{rel} = -\sqrt{\frac{2P}{\rho}}$$

$$\text{Thrust} = \dot{m} V_{rel} = \rho A V_{rel}^2 = 2PA$$

(a) For liftoff the thrust must equal the force of gravity. The total mass of the rocket is $M = M_r + M_w = 2.1\text{kg}$

$$Mg = 2PA$$

$$P = \frac{Mg}{2A} = 32800 \text{ Pa} = 4.76 \text{ lb/in}^2$$

(b) The rate water is ejected is

$$\gamma = \rho A v_{rel}$$

and the relative velocity

$$v_{rel} = \sqrt{\frac{2P}{\rho}}$$

$$= \sqrt{\frac{2(7\times10^{5}\,\mathrm{Pa})}{1000\ \mathrm{kg/m^3}}}$$

$$= \cancel{118\ \mathrm{m/s}}\quad 37\ \mathrm{m/s}$$

$$\gamma = \rho A v_{rel} = (1000\ \mathrm{kg/m^3})(3.14\times10^{-4}\,\mathrm{m^2})(118\ \mathrm{m/s})$$

$$= \overset{11.6}{\cancel{37}}\ \mathrm{kg/s}$$

The time for the rocket to expell all the mass is

$$M_w = t_e \gamma$$

$$t_e = \frac{M_w}{\gamma} = \frac{2\ \mathrm{kg}}{\cancel{37}\ 11.6\ \mathrm{kg/s}} = \cancel{0.054\ \mathrm{s}}$$

$$= 0.17\ \mathrm{s}$$

Let $u = M_r + M_w - \gamma t$

$du = -\gamma\, dt$

$$\dot{z} = -gt - \frac{2PA}{\gamma}\int_{u_0}^{u} \frac{du}{u}$$

$$= -gt - \frac{2PA}{\gamma} \ln u/u_0$$

$$= -gt - \frac{2PA}{\gamma} \ln\left[\frac{M_r + M_w - \gamma t}{M_r + M_w}\right]$$

(d) When water runs out $\gamma t_e = M_w$

$$\dot{z}(t_e) = -gt_e - \frac{2PA}{\gamma}\ln\left[\frac{M_r}{M_r + M_w}\right]$$

$$= -(9.81\ m/s^2)(\overset{0.17\,s}{\cancel{0.054\,s}}) - \frac{2(7\times10^{\cancel{6}\,5}\ N/m^2)(3.14\times10^{-4}\ m^2)}{\overset{11.6}{\cancel{37}}\ kg/s}\ln[\ ]$$

$$= \overset{-1.69\ m/s}{\cancel{-0.5294\ m/s}} + \overset{115\ m/s}{\cancel{362\ m/s}} = 113\ m/s$$

$$= \cancel{361\ m/s}$$

(Nice)

~~(e) How much higher?~~

$$\cancel{t_f = \frac{v_0}{g} = 36\,s}$$

$$\cancel{h = \frac{1}{2} g t^2 = 6300\,m}$$

(f) $\dot{z} = \dfrac{dz}{dt}$

$$z(t) = -\frac{1}{2}gt^2 + \int_0^t dt\, \frac{2PA}{\gamma} \ln\left[\frac{M_r + M_w - \gamma t}{M_r + M_w}\right]$$

How much higher?

$$t_f = \frac{v_0}{g} = \frac{113\,m/s}{9.81\,m/s^2} = 11.5\,s$$

$$h = \frac{1}{2} g t^2 = 660\,m$$

How high if 1 kg?

$$t_e = 0.085 \text{ s}$$

$$\dot{z}(t_e) \simeq 37.9 \text{ m/s}$$ TOO HIGH

How fast for 50 m?

$$t = \frac{250}{g} = \sqrt{\frac{100}{g}} \sim \sqrt{10} \sim 3 \text{ s}$$

$$v_0 = g t_f \simeq 30$$

$$\ln \frac{M_r}{M_w + M_r} = -1$$

$$\frac{M_r}{M_w + M_r} = \cancel{\ } e^{-1}$$

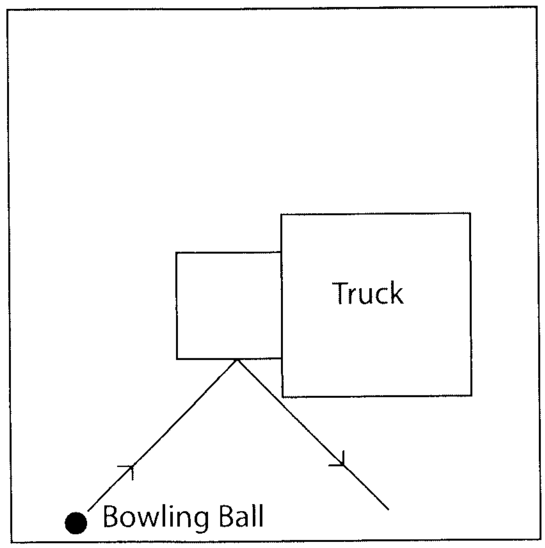
$$M_r = \cancel{\ } (M_w + M_r) \, 1/e$$

$$\cancel{(1+e) M_r} = \quad (e-1) M_r = M_w$$

$$M_w = 6.17\ kg$$

Take Home Problem

Problem 3.4 John Hubbard's dad still owns a dump truck. The truck is parked on an icy (frictionless) lake in Utah. For this problem the truck is modelled as two laminar squares as shown below. The smaller square has mass $M_1 = 1000\text{kg}$ and edge length $\ell_1 = 2\text{m}$. The larger square has mass $\frac{2}{3}M_1$ and edge length $\ell_2 = \frac{3}{2}\ell_1$. The truck is stationary on the ice. Josh Daily, still angry about the bonus points on the last test, shoots at the truck with a bowling ball bazooka. The bowling ball strikes the truck in the middle of the small rectangle (as shown) at an angle of 45° and rebounds at an equal angle of 45°. The ball travels horizontally. The ball rebounds with half the speed it had before it struck the truck. The initial speed of the bowling ball is $v_0 = 200\text{m/s}$. The mass of the bowling ball, m_b, is 5kg.



6 (a) Calculate the momentum and center of mass velocity of the truck after impact.

5 (b) Calculate the location of the center of mass of the truck. Draw this location on the diagram.

6 (c) Calculate the angular impulse delivered to the truck, that is the amount of angular momentum about the center of mass delivered to the truck during the impact.

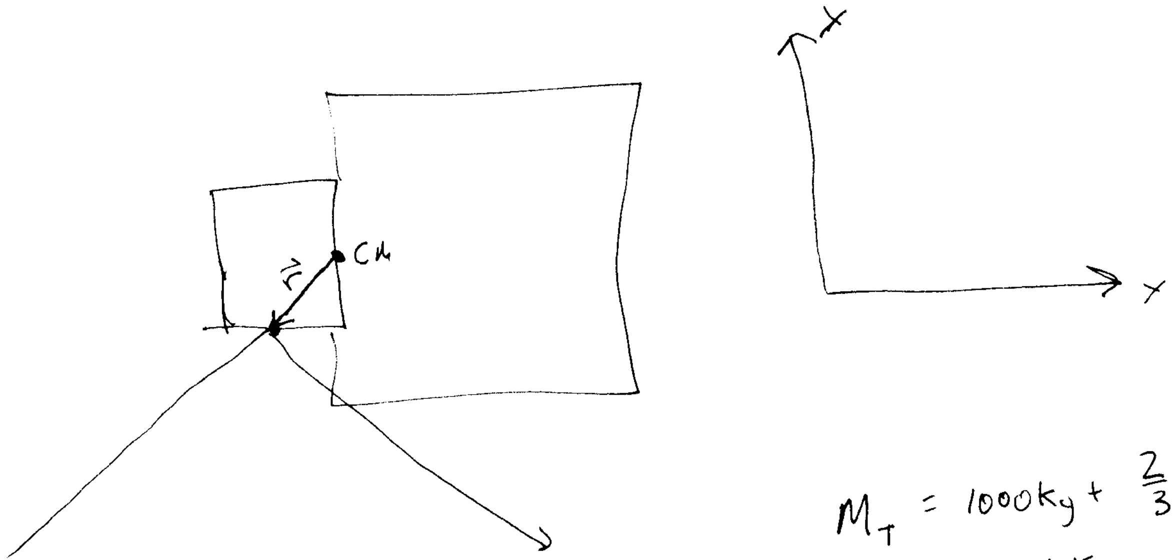
8 (d) Calculate the moment of inertia of the truck about its center of mass.

5 (e) Calculate the angular velocity and period of rotation of the truck after impact.

5 (f) Is the collision elastic?

5 (g) Compute Josh's velocity (assume $M_{Josh} = 60\text{kg}$) after firing.

3.4



$$M_T = 1000\,\text{kg} + \frac{2}{3}1000\,\text{kg}$$
$$= 1666\,\text{kg}$$

(a)

Cons

$$\vec{P}^{\,i}_{ball} = \vec{P}^{\,f}_{ball} + \vec{P}_{truck}$$

$$E^{i}_{ball} = E^{f}_{ball} + E^{f}_{truck}$$

$$\frac{m_{ball} V_0}{\sqrt{2}} = \frac{m_{ball} V_0}{2\sqrt{2}} + P_{tx}$$

$$m_{ball}\frac{V_0}{\sqrt{2}} = -\frac{m_{ball} V_0}{2\sqrt{2}} + P_{ty}$$

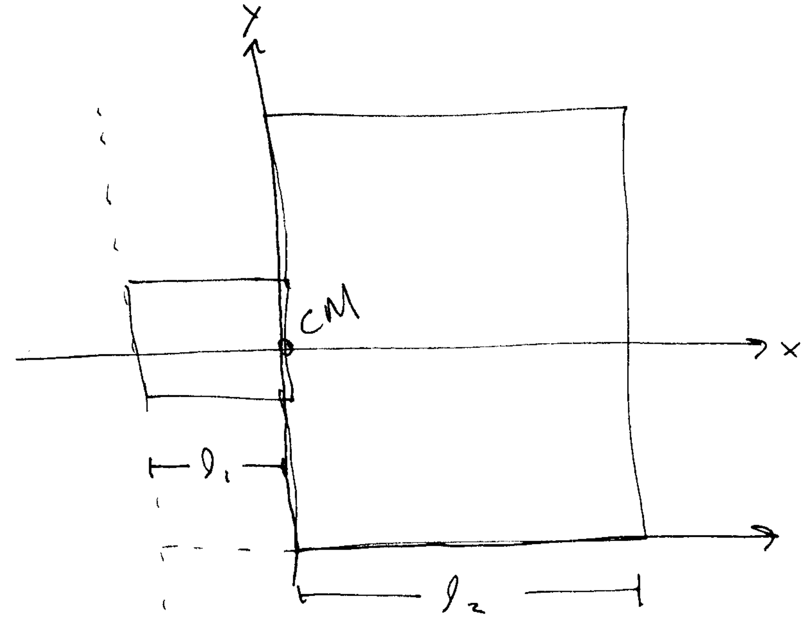
$$P_{t} = \frac{V_0 m_{ball}}{2\sqrt{2}} = 354\,\text{kg m/s}$$

$$P_{ty} = \frac{3}{2}\frac{V_0 m_{ball}}{\sqrt{2}} = 1060\,\text{kg m/s}$$

$$\vec{v}_{cm} = \frac{\vec{P}_c}{M} = \left(\cancel{0.177\,\text{m/s}}\ 0.212\,\text{m/s},\ \cancel{0.53\,\text{m/s}}\ 0.636\,\text{m/s},\ 0\right)$$

$$|v_{cm}| = 0.67\,\text{m/s}$$

(b)



$$l_2 = \tfrac{3}{2} l_1 \quad , \quad M_2 = \tfrac{2}{3} M_1$$

$$r_{cm} = \frac{1}{M_1 + M_2}\left[-\frac{l_1}{2} M_1 + \frac{l_2}{2} M_2 \right]$$

$$= \frac{1}{2(M_1+M_2)}\left[-l_1 M_1 + \left(\tfrac{3}{2} l_1\right)\left(\tfrac{2}{3} M_1\right) \right]$$

$$= 0$$

(c)

$$\Delta \vec{p} = \frac{v_0 m_b}{2\sqrt{2}} (1, 3)$$

$$\vec{r}_{cm,\, impact} = -\frac{l_1}{2} (1, 1)$$

$$\Delta \vec{L} = \vec{r} \times \Delta \vec{p} = -\frac{v_0 m_b l_1}{4\sqrt{2}} (1,1) \times (1,3)$$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & 3 & 0 \end{vmatrix} = 3 - 1\, \hat{k} = 2\hat{k}$$

$$\Delta \vec{L} = \frac{-v_0 m_b l_1}{2\sqrt{2}} \hat{k}$$

$$= 707\ \mathrm{kg}\,\frac{m^2}{s}\, \hat{k}$$

(d) Calculate the moment of inertia about cm,

The moment of inertia of a square lamina is

$$I = \frac{2a^2}{12} = \frac{d^2}{6}$$

The moment of the small rectangle about the CM

is $$I_1 = M_1 \frac{l_1^2}{6} + M_1\left(\frac{l_1}{2}\right)^2 \qquad \text{(Parallel axis thm)}$$

$$= \frac{M_1 l_1^2}{2}\left[\frac{1}{3} + \frac{1}{2}\right]$$

$$= \frac{5 M_1 l_1^2}{12}$$

The moment of the large rectangle about cm

is $$I_2 = M_2 \frac{l_2^2}{6} + M_2\left(\frac{l_2}{2}\right)^2$$

$$= \frac{5}{12} M_2 l_2^2 = \frac{5}{12}\left(\frac{2}{3}M_1\right)\left(\frac{3}{2}l_1\right)^2$$

$$= \frac{5}{8} M_1 l_1^2$$

Total $$I_{cm} = I_1 + I_2 = \frac{5 M_1 l_1^2}{4}\left(\frac{1}{2} + \frac{1}{3}\right) = \frac{25}{24} M_1 l_1^2 = \frac{25}{6} M_1$$

$$I_{cm} = \frac{25}{24}\,(1000\,kg)\,(2m)^2$$

$$= \frac{25}{6}\times 10^3\ kgm^2 = 4.16\times 10^3\ kgm^2$$

(d) $\Delta L = I_{cm}\,\Delta\omega$

$$= \frac{500\sqrt{2}}{(1000)\,25/6} = \frac{3\sqrt{2}}{25}\ s^{-1}$$

$$\omega = \frac{\Delta L}{I_{cm}} = \frac{707\ kgm^2/s}{4.16\times 10^3\ kgm^2} = 0.17\ s^{-1}$$

$$= \frac{2\pi}{T}$$

$2\pi/\omega = T = 37\,s$ Rotate once every $37\,s$

(e)

Elastic if no energy lost in collision

$$E_{before} = \frac{1}{2} m_b V_0^2$$

$$= \frac{1}{2} 5\,kg \left(200\,m/s\right)^2$$

$$= 100{,}000\,J$$

$$E_{after} = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I \omega^2 + \frac{1}{2} m_{ball} \left(\frac{V_0}{2}\right)^2$$

$$= \frac{P_{cm}^2}{2M} + \frac{1}{2} I \omega^2 + \frac{1}{2} m_{ball} \left(\frac{V_0}{2}\right)^2$$

$$= \frac{V_0^2 m_b^2 \left(1 + 3^2\right)}{16(M_1 + M_2)} + \frac{1}{2}\left(4160\,kg\,m^2\right)\left(0.17\,s^{-1}\right)^2 + 25{,}000\,J$$

$$= \frac{\left(200\,m/s\right)^2 \left(5\,kg\right)^2 \left(10\right)}{16\left(1000 + \frac{2}{3} 1000\right)} + 60\,J + 25{,}000\,J$$

$$= 25{,}435\,J$$

Highly inelastic $\Rightarrow$ lots of damage to something.

(f) Conserve Josh's momentum

$$0 = p_{josh} + p_{ball}$$

$$= M_{josh} V_{josh} + m_{ball} v_{ball}$$

$$V_{josh} = -\frac{m_{ball} V_{ball}}{M_{josh}} = 17 \text{ m/s}$$