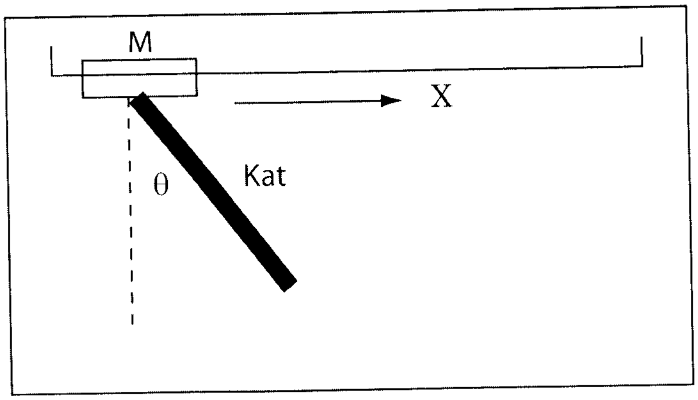
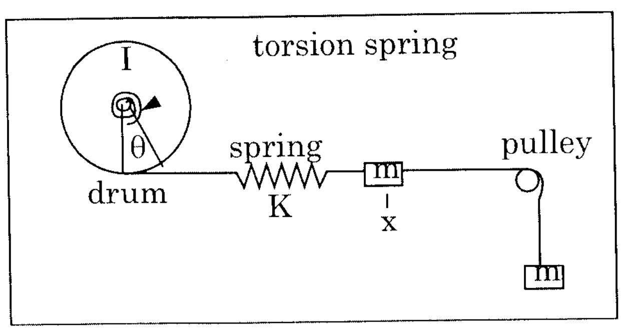
Mechanics Spring 2003 - Test 4

I have to grade these, so your solution will be well presented just in case I spent the entire week dealing with UPII students whining about points.

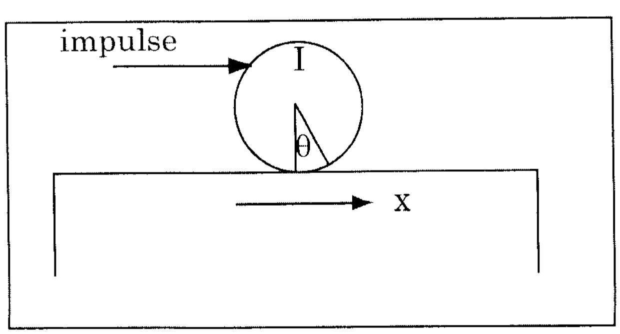
Problem 4.1 The Kat (my youngest daughter) is well modelled as a rigid rod of length ℓ and mass m_{kat}. She jumps on a playground toy that slides horizontally. The playground slider has mass M. Kat pivots about the point she holds the slider. Write the equations of motion (Using the Lagrange formalism) for the location of the slider X and the angle at which Kat hangs θ. The moment of inertia of a rigid rod about its center is $\frac{ma^2}{12}$ where a is the length of the rod. You do not need to solve the equations of motion.



Problem 4.2 The system below has two masses each of mass m and a drum with moment of inertia (about its center) of I. There is a torsion spring connected to the axis of the drum that restores the drum to its equilibrium location $\theta = 0$. The potential energy of the drum/torsion spring combination is $\frac{1}{2}\Gamma\theta^2$ where Γ is a constant. The drum is connected to a mass m through a spring with spring constant K. That mass is connected to another mass m by a massless cord of constant length over a massless pulley. Gravity acts downward on the second mass. The first mass moves only in the x direction, so its location can be described by the generalized coordinate x. The system is in equilibrium when $\theta = 0$ and $x = 0$. Compute the equations of motion of the system. Bonus: Ignoring the constant term, write the matrix whose determinant would have to be zero to find the normal frequencies of the system.



Problem 4.3 This problem does not require the Lagrangian method. A billiard ball is struck with an impulsive force directed parallel to the flat horizontal surface on which the ball sits. A linear momentum of $\Delta\vec{p} = p_0\hat{x}$ is transferred to the ball at a point 3/4 of the height of the ball, as drawn. The table is rough (felted in fact) and has a coefficient of kinetic friction, μ_k. You may assume the balls slips while it rolls. Write the trajectory of the ball ($x(t)$ and $\theta(t)$) as a function of time while the ball is slipping.



Fake Bonus Problem What is the color of Jon Hubbard's dad's dump truck?

Test Notes

4.3 Let the radius of the ball be a.
and the mass be m.

The moment of inertia is $\frac{2}{5} m a^2$

Rotating too fast to meet condition of rolling.

4.2 Positive θ as drawn. Radius of drum a.

(4.1) The kinetic energy of the system is

$$T = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m_{tot} \left(\dot{x}_{cm}^2 + \dot{y}_{cm}^2 \right) + \frac{1}{2} I_{cm} \dot{\theta}^2$$

and the potential energy

$$V = -\frac{mg\,l}{2} \cos\theta \qquad \left[0 \text{ at } \theta = \pi/2 \text{, the support} \right]$$

<u>Work out the CM</u>

$$x_{cm} = X + l/2 \sin\theta$$

$$y_{cm} = -l/2 \cos\theta$$

$$\dot{x}_{cm} = \dot{X} + l/2 \cos\theta\, \dot{\theta}$$

$$\dot{y}_{cm} = + l/2 \sin\theta\, \dot{\theta}$$

$$\dot{x}_{cm}^2 + \dot{y}_{cm}^2 = \dot{X}^2 + l \cos\theta\, \dot{\theta} \dot{X} + \frac{l^2}{4} \dot{\theta}^2$$

$$L = T - V = \frac{1}{2}(M + m_{kat})\dot{x}^2 + \frac{1}{2} m_{kat}\, l\, \dot{\theta}\dot{x}\cos\theta$$

$$+ \frac{1}{2}\left(\frac{m_{kat} l^2}{4} + I_{cm}\right)\dot{\theta}^2 + \frac{mgl}{2}\cos\theta$$

<u>x eqn</u>

$$\frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial \dot{x}} = (M + m_{kat})\dot{x}$$

$$+ \frac{1}{2} m_{kat}\, l\, \dot{\theta}\cos\theta$$

$$\boxed{+ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 = (M + m_{kat})\ddot{x} + \frac{1}{2} m_{kat}\, l\, \ddot{\theta}\cos\theta - \frac{1}{2} m_{kat}\, l\, \dot{\theta}^2 \sin\theta}$$

<u>θ eqn</u>

$$\frac{\partial L}{\partial \theta} = -\frac{1}{2} m_{kat}\, l\, \dot{\theta}\dot{x}\sin\theta - \frac{mgl}{2}\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m_{kat}\, l\, \dot{x}\cos\theta + \left(\frac{ml^2}{4} + I_{cm}\right)\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m_{kat}\, l\, \ddot{x}\cos\theta - \frac{1}{2} m_{kat}\, l\, \dot{x}\dot{\theta}\sin\theta$$

$$+ \left(\frac{ml^2}{4} + I_{cm}\right)\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = -\frac{1}{2} m_{rot} l \dot\theta \dot X \sin\theta - \frac{mgl}{2}\sin\theta$$

$$- \frac{1}{2} m_{rot} l \ddot X \cos\theta + \frac{1}{2} m_{rot} l \dot X \dot\theta \sin\theta$$

$$- \left(\frac{ml^2}{4} + I_{cm}\right)\ddot\theta = 0$$

$$-\frac{mgl}{2}\sin\theta - \frac{1}{2} m_{rot} l \ddot X \cos\theta - \left(\frac{ml^2}{4} + I_{cm}\right)\ddot\theta = 0$$

Key (35 pts)

<u>Analysis</u> 20 pts

+5 V

+5 T_{rot}

+5 T_{slider}

+5 x_{cm}, y_{cm}

Math 15 pts - Differentiation

-3 pts if I about end.

4.2 $T = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2}(2m)\dot{x}^2$

Positive θ same direction as positive x

$$V = \frac{1}{2}\Gamma\theta^2 + \frac{1}{2}k(x - a\theta)^2 - mgx$$

$$L = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2}(2m)\dot{x}^2 - \frac{1}{2}\Gamma\theta^2 - \frac{1}{2}k(x-a\theta)^2 + mgx$$

x - eqn

$$\frac{\partial L}{\partial x} = -k(x - a\theta) + mg \qquad \frac{\partial L}{\partial \dot{x}} = 2m\dot{x}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 2m\ddot{x}$$

$$\boxed{\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = -k(x-a\theta) + mg - 2m\ddot{x} = 0}$$

θ eqn

$$\frac{\partial L}{\partial \theta} = -\Gamma\theta + ka(x - a\theta) \qquad \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = I\ddot{\theta}$$

$$\boxed{\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -\Gamma\theta + ka(x - a\theta) - I\ddot{\theta} = 0}$$

Bonus

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix} + \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} = 0$$

$$\begin{pmatrix} -k & ka \\ ka & -ka^2 \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix} + \begin{pmatrix} 2m & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} = 0$$

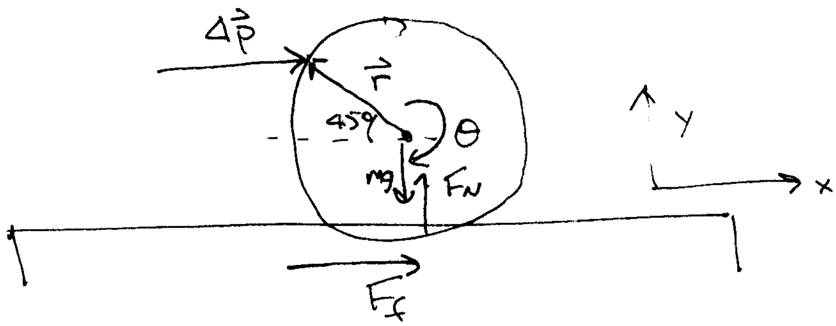
For normal mode,

$$\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}\cos(\omega t+\delta)$$

$$\begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} = -\omega^2\begin{pmatrix} d_1 \\ d_2 \end{pmatrix}\cos(\omega t+\delta)$$

$$\begin{pmatrix} -k - 2m\omega^2 & ka \\ ka & -ka^2 - I\omega^2 \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix} = 0$$

4.3



$$\vec{\Delta L} = \vec{r} \times \Delta\vec{p} = a\,p_o \sin(\cancel{135}) \hat{k}$$

$$= \frac{a\,p_o}{\sqrt{2}} \hat{k}$$

I made a classic mistake

see next page

The strike transfers linear and angular moments causing an initial translation $v_o = \frac{p_o}{m}$

and an initial rotation rate $|\Delta L| = I\omega_o$

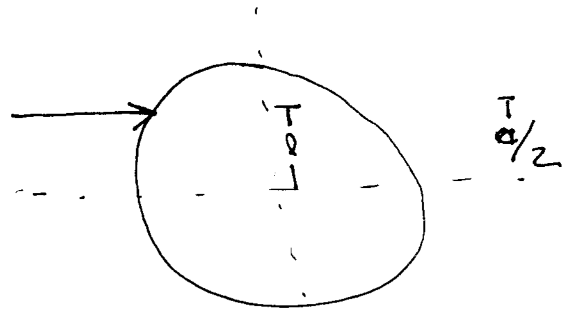
$$\omega_o = \frac{a\,p_o}{\sqrt{2}\, I}$$

The ball slips so the force of friction acts to slow rotation.

The force of friction is $F_f = \mu_k F_N = \mu_k mg$

The correct way to compute the angular momentum transfer.

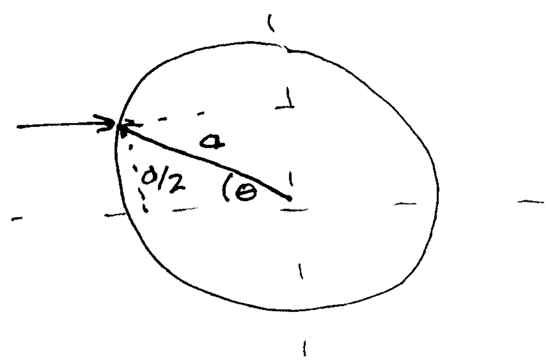
(A)



The strike is $\perp$ to center

$$\Delta L = l p_0 = \frac{a p_0}{2}$$

(B) Do the geometry



$$\sin\theta = \frac{a/2}{a} = 1/2$$

$$|\Delta L| = |\vec{r} \times \Delta\vec{p}| = |\vec{r}||\Delta p| \sin\theta$$

$$= \frac{a p_0}{2}$$

The component equations of motion are

x - component

$$m\ddot{x} = +\mu_K m g$$

$$\ddot{x} = \mu_K g$$

<u>Torque</u>

$$N = F_f a = -I\ddot{\theta}$$

$$\ddot{\theta} = -\frac{\mu_K m g a}{I}$$

$$\boxed{\begin{aligned} \theta(t) &= \omega_0 t - \frac{1}{2}\left(\frac{\mu_K m g a}{I}\right) t^2 \\ x(t) &= v_0 t + \frac{1}{2}\mu_K g t^2 \end{aligned}}$$