Test Notes

ln(1) = C

# Mechanics Spring 2003 - Test 1

14 pts **Problem 1.1** A cyclist (could be Greg LeMonde, Caleb), rides around an unbanked (flat) circular track. The cyclist completes each lap in time $T_{lap}$, and thus moves at a constant angular velocity. As the race goes on, the rider is gradually forced from an initial radius $r_0$ to a radius $\Delta r + r_0$, where $\Delta r$ is a constant. The radius of the cyclist's trajectory is given by

$$r(t) = r_0 + \Delta r\left(1 - e^{-\frac{t}{\tau}}\right)$$

where $\tau$ is constant that captures the characteristic time to change radii.

2 **(a)** Write $\vec{r}(t)$.

4 **(b)** Calculate $\vec{v}(t)$.

2 **(c)** Calculate the kinetic energy as a function of time.

4 **(d)** Calculate $\vec{a}(t)$.

3 ~~4~~ **(e)** Calculate the angle between the acceleration and velocity as a function of time. This may be messy.

* 17 **Problem 1.2** A particle of mass $m$ is shot into a region where it experiences a force $F = -\alpha v e^{-\beta v}$ where $\alpha$ and $\beta$ are positive constants. Take $x(0) = 0$ and $v(0) = v_0$ as the initial conditions.

2 **(a)** Does a potential function exist for this force? Why or why not?

4 **(b)** Find the velocity as a function of the distance $x$ from the point the particle enters the force field.

4 **(c)** Find the distance the particle travels before coming to a stop.

3 **(d)** Write the integral, with appropriate limits of integration, that you would evaluate to find the trajectory, $x(t)$, of the particle.

2 **(e)** Write kinetic energy of the particle as a function of position.

2 **(f)** How much total energy is dissipated by the force before the particle comes to rest?

15 **Problem 1.3** Consider the potential function $U(x) = ax^3 - bx$, where $a$ and $b$ are positive constants. It may help to sketch the potential.

3 **(a)** Compute the location of the local minima, $x_{min}$ and the local maxima $x_{max}$ of this potential.

3 **(b)** What condition must ~~the total energy, $E_{sys}$~~ $T(x_{min})$, of a particle of mass $m$ satisfy so that the particle oscillates about $x_{min}$, that is the particle is confined to potential well about $x_{min}$?

4 (c) Write the velocity as a function of position for a particle of mass $m$ with initial velocity $v_0$ at $x_{min}$ at $t = 0$.

3 (d) If a particle of mass $m$ was released from $x = 0$ with zero initial velocity, compute its other turning point.

2 (e) Compute the natural frequency of this particle from small amplitude oscillations about $x_{min}$.

(26) **Problem 1.4** My daughter Katherine(Kat) likes to jump on a trampoline. This problems asks you to analyze the Kat/trampoline system. Let the location of the center of the trampoline when Kat stands still be $x = 0$ and upward be positive. The force of gravity simply shifts the equilibrium position and may be ignored in the analysis of the oscillations. I asked Kat to jump once. Using a tape measure and my wrist watch, I measure the amplitude of the first maxima to be 2 inches and the amplitude of the second maxima 1 inch. Kat oscillated up and down for a while and came to a stop. The period of these oscillations is $\frac{2}{3}$ seconds.

1 (a) From the available information, what color is my youngest daughter's hair?

2 (b) Is the motion of the trampoline/Kat system overdamped, critically damped, or underdamped? Support your choice. Tell what you would expect to see if the cases you did not choose were the case.

2 (c) Calculate the angular damping frequency, $\omega_d$.

4 (d) Calculate the damping constant $\gamma$.

2 (e) Calculate the natural frequency $\omega_0$.

4 (f) If the initial maxima happens at $t = 0$. Use initial conditions $x_0 = -2$ inches and $v_0 = 0$ at $t = 0$. Write the trajectory of the trampoline surface $x(t)$.

2 (g) Compute the resonant frequency of the trampoline.

Kat begins jumping once per second, $T = 1$s, applying a sinusoidal driving force $F = F_0 \cos \omega t$, where $\omega = 2\pi/T$ and we will assume $F_0 = mg$ where Kat's mass is $m = 30$kg.

3 (h) Compute the amplitude of the trampoline/Kat system under this driving force.

3 (i) Compute the phase shift of the trampoline/Kat system under this driving force.



Bonus Draw electric field of point charge

(1.1)

(a) $$\vec{r}(t) = r(t)\,\hat{e}_r$$

$$= \left[r_0 + \Delta r\left(1 - e^{-t/\tau}\right)\right]\hat{e}_r$$

(b) $$\dot{r} = \frac{\Delta r}{\tau} e^{-t/\tau} \qquad \ddot{r} = -\frac{\Delta r}{\tau^2} e^{-t/\tau}$$

$$\dot{\theta} = \omega = \frac{2\pi}{T_{lap}} \qquad \ddot{\theta} = 0$$

$$\vec{v}(t) = \dot{r}\,\hat{e}_r + r\dot{\theta}\,e_\theta$$

$$= \left(\frac{\Delta r}{\tau} e^{-t/\tau}\right)\hat{e}_r + \omega\left[r_0 + \Delta r\left(1 - e^{-t/\tau}\right)\right]\hat{e}_\theta$$

(c) $$T = \frac{1}{2} m\vec{v}\cdot\vec{v}$$

$$\vec{v}\cdot\vec{v} = \left(\frac{\Delta r}{\tau}\right)^2 e^{-2t/\tau} + \omega^2\left[r_0 + \Delta r\left(1 - e^{-t/\tau}\right)\right]^2$$

(d) $$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (\underset{0}{\underline{r\ddot{\theta}}} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$= \left[-\frac{\Delta r}{\tau^2} e^{-t/\tau} - \omega^2\left[r_0 + \Delta r\left(1 - e^{-t/\tau}\right)\right]\right]\hat{e}_r$$

$$+ \frac{2\omega\Delta r}{\tau} e^{-t/\tau}\,\hat{e}_\theta$$

(e) $$\cos\theta = \frac{\vec{a}\cdot\vec{v}}{a\,v}$$

$$a = \left[\left(\frac{\Delta r}{\tau^2} e^{-t/\tau} + \omega^2\left[r_0 + \Delta r\left(1 - e^{-t/\tau}\right)\right]\right)^2 + \frac{4\omega^2\Delta r^2}{\tau^2} e^{-2t/\tau}\right]^{1/2}$$

$v$ above

$$\vec{a}\cdot\vec{v} = \left[-\frac{\Delta r}{\tau^2} e^{-t/\tau} - \omega^2\left[r_0 + \Delta r\left(1 - e^{-t/\tau}\right)\right]\right]\frac{\Delta r}{\tau} e^{-t/\tau}$$

$$+ \omega\left(\frac{2\omega\Delta r}{\tau}\right)\left(r_0 + \Delta r\left(1 - e^{-t/\tau}\right)\right)$$

Problem 1.2

(a) The force is velocity & dependent so no potential exists.

(b) $F = -\alpha v e^{-\beta v} = m v \frac{dv}{dx}$

&

$$\int_0^x -\frac{\alpha}{m}\, dx = \int_{v_0}^{v} e^{\beta v}\, dv$$

$$-\frac{\alpha x}{m} = \frac{1}{\beta}\left[e^{+\beta v}\right]_{v_0}^{v} = \frac{1}{\beta}\left(e^{\beta v} - e^{\beta v_0}\right)$$

$$e^{\beta v_0} - \frac{\alpha \beta x}{m} = e^{\beta v}$$

$$v = \frac{1}{\beta} \ln\left[e^{\beta v_0} - \frac{\alpha \beta x}{m}\right]$$

(c) Range occurs when $v = 0$, $\ln(1) = 0$

&

$$e^{\beta v_0} - \frac{\alpha \beta x}{m} = 1$$

$$e^{\beta v_0} - 1 = \frac{\alpha \beta x}{m} \qquad x_r = \frac{m}{\alpha \beta}\left(e^{\beta v_0} - 1\right)$$

(d) $$v = \frac{dx}{dt}$$

3 $$\int_0^t dt = \int_0^x \frac{dx}{v(x)} = \int_0^x \frac{\beta\, dx}{\ln\left(e^{\beta v_0} - \frac{\alpha \beta x}{m}\right)}$$

(e) $$T = \frac{1}{2} m v^2 = \frac{m}{2\beta^2}\left(\ln\left[e^{\beta v_0} - \frac{\alpha \beta x}{m}\right]\right)^2$$

2

(f) All the energy is dissipated so

2 $$E_{diss} = T_{initial} = \frac{1}{2} m v_0^2$$

1.3

(a) Extrema occur at

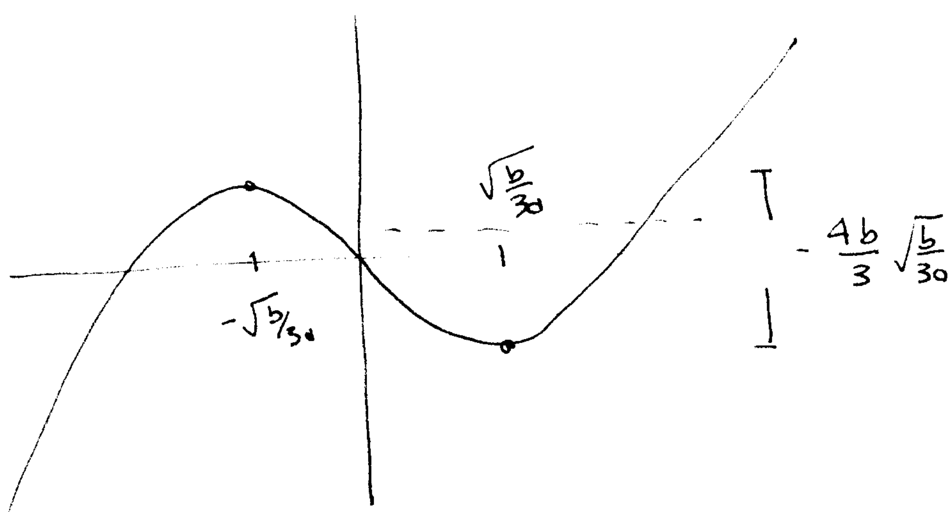
$$\frac{dU}{dx} = 0 = 3ax^2 - b$$

3

$$x = \pm \sqrt{\frac{b}{3a}}$$

The positive root is a minima and the negative root a maxima.

$$x_{min} = \sqrt{\frac{b}{3a}} \qquad x_{max} = -\sqrt{\frac{b}{3a}}$$



(b) To oscillate back and forth the ~~to~~ kinetic energy $T(x_{min})$ must be less than barrier height

$$U(x_{max}) - U(x_{min}) > T(x_{min})$$

3

$$U(x_{max}) = -a\left(\frac{b}{3a}\right)\sqrt{\frac{b}{3a}} + b\sqrt{\frac{b}{3a}}$$

$$U(x_{min}) = a\left(\frac{b}{3a}\right)\sqrt{\frac{b}{3a}} - b\sqrt{\frac{b}{3a}} =$$

$$U(x_{max}) - U(x_{min}) = 2\sqrt{\frac{b}{3a}}\left[\frac{2b}{3}\right] = \frac{4b}{3}\sqrt{\frac{b}{3a}} > T_0$$

(c) $$E_{sys} = T_0 + U(x_{min}) = \frac{1}{2}mv^2 + U(x)$$

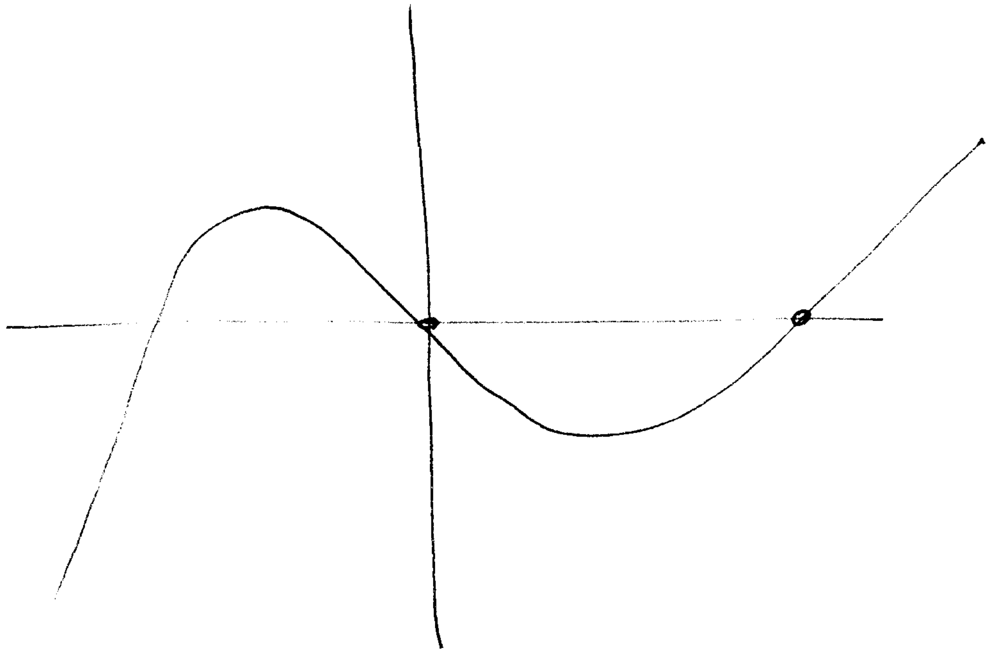
$$v(x) = \left[\frac{2}{m}\left(E_{sys} - U(x)\right)\right]^{1/2}$$

4

~~$\left[\frac{2}{m}\left(E_{sys} - U(x)\right)\right.$~~

$$v(x) = \left[\frac{2}{m}\left(\left(\frac{1}{2}mv_0^2 - \frac{2b}{3}\sqrt{\frac{b}{3a}}\right) - ax^3 + bx\right)\right]^{1/2}$$

(d)



Released $E_{sys} = U(0) = 0$

Other turning point also happens at $x = 0$

$$U(x) = 0 = ax^3 - bx = 0$$

$$x^2 = \frac{b}{a}$$

3

$$x = \sqrt{b/a}$$

(e) The natural frequency is

$$\omega_0 = \sqrt{\frac{k}{m}}$$

2

$$k = \left.\frac{d^2U}{dx^2}\right|_{x_{min}}$$

$$\frac{d^2U}{dx^2} = 6ax \qquad \left.\frac{d^2U}{dx^2}\right|_{x_{min}} = 6a\left(\sqrt{\frac{b}{3a}}\right)$$

$$\omega_0^2 = \frac{6a}{m}\sqrt{\frac{b}{3a}}$$

1.4

$$T_d = \frac{2}{3}\,s \qquad A_1 = 2\,in$$

$$A_2 = 1\,in$$

$$\omega_d = \frac{2\pi}{T_d} = (3\pi)\,s^{-1}$$

(a) Blonde

(b) underdamped because it oscillates

(c) $\omega_d = (3\pi)\,s^{-1} = 9.42\,s^{-1}$

(d) $$\frac{A_2}{A_1} = \frac{1}{2} = e^{-\gamma T_d}$$

$$\ln\left(\tfrac{1}{2}\right) = -\gamma T_d$$

$$\gamma = -\frac{1}{T_d}\ln\left(\tfrac{1}{2}\right) = -\frac{3}{2}(s^{-1})\ln\left(\tfrac{1}{2}\right)$$

$$= 1.04\,s^{-1}$$

(e) $$\omega_d^2 = \omega_0^2 - \gamma^2$$

$$\omega_0 = \sqrt{\omega_d^2 + \gamma^2} = 9.48\,s^{-1}$$

$3.02\pi$

(f) $x(t) = A\cos(\omega_d t + \phi)e^{-\gamma t}$

$2\text{in} = x(0) = A_0 = A\cos\phi \Rightarrow$

$\dot{x}(t) = -\gamma A e^{-\gamma t}\cos(\omega_d t + \phi)$

$- A e^{-\gamma t}\omega_d \sin(\omega_d t + \phi)$

$\dot{x}(0) = 0 = -\gamma A\cos(\phi) - A\omega_d \sin(\phi)$

~~Solve for A~~ ; $A = \dfrac{A_0}{\cos\phi}$

$\tan\phi = -\dfrac{\gamma}{\omega_d}$ $-6.26$

23.7 in

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$x(t) = A_1 e^{-\gamma t}\cos\omega_d t + A_2 e^{-\gamma t}\sin\omega_d t$

$x(0) = A_1 = A_0 = -2\text{in}$

$$\dot{x}(t) = -A_1\gamma e^{-\gamma t}\cos\omega_d t - A_1\omega_d e^{-\gamma t}\sin\omega_d t$$
$$- A_2\gamma e^{-\gamma t}\sin\omega_d t + A_2\omega_d e^{-\gamma t}\cos\omega_d t$$

$$\dot{x}(0) = 0 = -A_1\gamma + A_2\omega_d$$

$$A_2 = \frac{A_1\gamma}{\omega_d} = \frac{-2\text{in}\,(1.04\,s^{-1})}{9.42\,s^{-1}}$$

$$= -0.22\text{in}$$

$$x(t) = -2(\text{in})\,e^{-\gamma t}\cos\omega_d t - 0.22\text{in}\, e^{-\gamma t}\sin\omega_d t$$

(g) $$\omega_r^2 = \omega_d^2 - \gamma^2 = \omega_0^2 - 2\gamma^2$$
$$\omega_r = \sqrt{(3\pi\,s^{-1})^2 - (1.04\,s^{-1})^2} = 9.37\,s^{-1}$$

(h) $$\omega = \frac{2\pi}{1s} = 2\pi\,s^{-1}$$

$$A(\omega) = \frac{g}{\left((\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right)^{1/2}}$$

$$A(\omega) = \frac{9.81\ m/s^2}{\left(\underbrace{(9.48^2 - 2\pi^2)^2}_{2539} + \underbrace{4(1.04)^2(2\pi)^2}_{170}\right)^{1/2}}$$

$$= 0.19\ m = 20cm$$

7.4 inches

(8.8) / ...

## (i) Phase shift

$$\tan\phi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2} = .259$$

9.48

$\phi =$ ~~.253~~

$\varphi = 14.5°$

.25 rad