Laminar Motion of Rigid Bodies

Rigid Body - A system of particles where the relative position of each particle is fixed.

Center of Mass - Average location of the mass of the system of total mass M.

$$\vec{r}_{cm} = \frac{1}{M} \sum m_i \vec{r}_i$$

$$= \frac{1}{M} \int \vec{r}\, dm$$

$$dm = \rho\, dV \qquad \rho \text{ - volume mass density}$$

$$= \sigma\, dA \qquad \sigma \text{ - surface mass density}$$

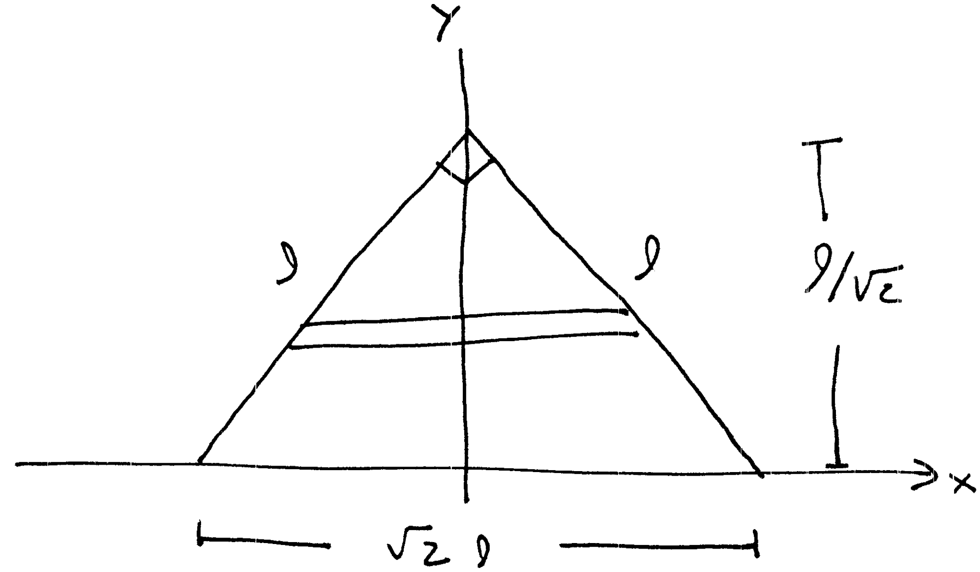
$$= \lambda\, ds \qquad \lambda \text{ - linear mass density}$$

$\Rightarrow$ If a body has a plane of symmetry the CM lies on the plane.

$\Rightarrow$ If body composite,

$$\vec{r}_{cm} = \frac{1}{M} \left[m_1 \vec{r}_1^{\,cm} + m_2 \vec{r}_2^{\,cm} + \cdots \right]$$

<u>Example</u> Find CM of right triangular lamina with constant mass density σ.



- Total mass $M = \sigma A = \sigma\left(\frac{1}{2}\, \text{base} \cdot \text{height}\right)$

$$= \frac{1}{2}\sigma\left(\sqrt{2}\, l\right) l/\sqrt{2}$$

$$= \frac{1}{2}\sigma l^2$$

- By symmetry the CM lies on the y axis.

$$\Rightarrow x_{cm} = 0, \quad z_{cm} = 0$$

- $$y_{cm} = \frac{1}{M}\int y\, dm$$

The line forming the right side of the triangle is

$$y = \ell/\sqrt{2} - x$$

$$x = \ell/\sqrt{2} - y$$

The length of a strip of the lamina is

$$2x = 2\left(\ell/\sqrt{2} - y\right) \equiv L$$

The mass of a strip is $\sigma L \, dy$

$$dm = 2\sigma\left(\ell/\sqrt{2} - y\right)dy$$

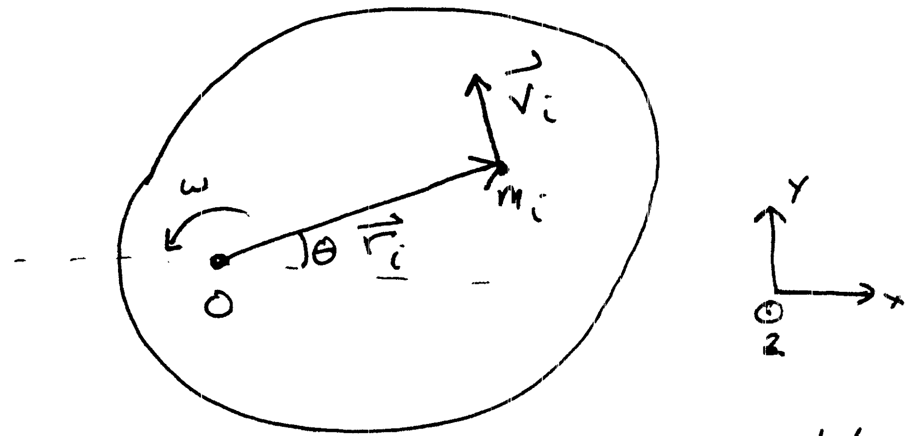
The center of mass is then

$$y_{cm} = \frac{1}{M}\int y\,dm = \frac{1}{M}\int_0^{\ell/\sqrt{2}} 2\sigma y\left(\ell/\sqrt{2} - y\right)dy$$

$$= \frac{2\sigma\ell^3}{M}\,\frac{1}{12\sqrt{2}}$$ after some annoying calculus

$$y_{cm} = \frac{\ell}{3\sqrt{2}}$$

Rotation of Rigid Body about Fixed Axis



If the body is rigid and constrained to rotate about fixed axis O, all particles move in circles with angular velocity $\dot{\theta}$.

$$\vec{r}_i = r_i \cos\theta\, \hat{x} + r_i \sin\theta\, \hat{y}$$

r_i constant

$$\vec{v}_i = -r_i \sin\theta\, \dot{\theta}\, \hat{x} + r_i \cos\theta\, \dot{\theta}\, \hat{y}$$

Define axis of rotation by vector $\vec{\omega} = \omega\hat{z} = \dot{\theta}\hat{z}$

$\Rightarrow$ Direction of rotation given by right hand rule. Point ~~[illegible]~~ thumb in direction of $\vec{\omega}$, finger curl in direction of rotation.

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \dot{\theta} \\ r_i \cos\theta & r_i \sin\theta & 0 \end{vmatrix}$$

$$= -r_i \dot{\theta} \sin\theta \, \hat{x} + r_i \dot{\theta} \cos\theta \, \hat{y}$$

Kinetic Energy of Rotation

$$T_{rot} = \frac{1}{2} \sum m_i \vec{v}_i \cdot \vec{v}_i$$

$$= \frac{1}{2} \sum m_i (\vec{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \cdot \vec{r}_i)$$

$$= \frac{1}{2} \sum m_i \, |\vec{\omega} \times \vec{r}_i|^2$$

$$= \frac{1}{2} \sum m_i (\omega r_i \sin\alpha)^2$$

$$= \frac{1}{2} \sum m_i r_i^2 \omega^2$$

where α is the angle between $\vec{r}_i$ and $\vec{\omega}$

but $\alpha = 90°$.

Dfn <u>Moment of Inertia</u> (about rotation axis O)

Let rotation axis be z axis

$$I_z = \sum m_i r_i^2 = \sum m_i (x_i^2 + y_i^2)$$

<u>Angular Momentum</u> (about z axis)

$$\vec{L} = \sum \vec{L}_i = \sum m_i (\vec{r}_i \times \vec{v}_i)$$

$$\vec{L}_i = m_i \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_i & y_i & 0 \\ \dot{x}_i & \dot{y}_i & 0 \end{vmatrix}$$

$$= m_i (x_i \dot{y}_i - \dot{x}_i y_i)\, \hat{z}$$

Use $\vec{r}_i$, $\vec{v}_i$ from two pages ago

$$\vec{L}_i = m_i \left((r_i \cos\theta)(r_i \cos\theta\, \dot{\theta}) - (-r_i \sin\theta)(r_i \sin\theta\, \dot{\theta}) \right) \hat{z}$$

$$= m_i r_i^2 \omega$$

The total angular momentum about the fixed axis

$$\vec{L} = \sum \vec{L}_i = \sum m_i r_i^2 \omega \hat{z} = I_z \omega \hat{z}$$

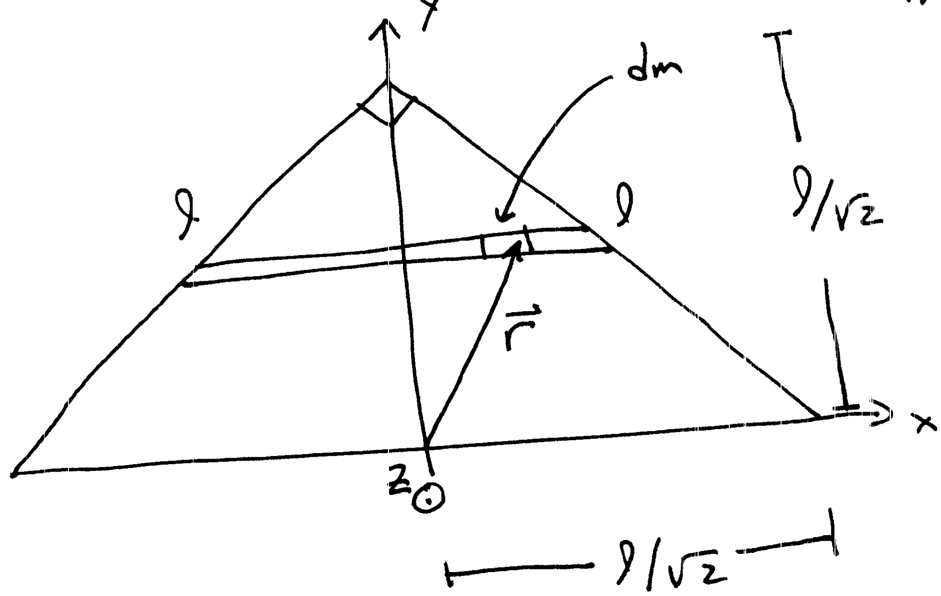
The torque about the fixed axis

$$N_z = \frac{dL}{dt} = I_z \dot{\omega}$$

More Moments

$I = \int dm\, r^2$ gives moment of inertia about any axis where r^2 is the $\perp$ distance from the axis.

Ex Return to our isoceles right triangle → Compute moment about z axis



Cut triangle into strips

$$dm = \sigma\, dx\, dy$$

$$r^2 = x^2 + y^2$$

$$I = \int r^2 dm = \sigma \int_0^{l/\sqrt{2}} dy \int_{-(l/\sqrt{2} - y)}^{l/\sqrt{2} - y} dx\, (x^2 + y^2)$$

Perpendicular Axis Thm   For a thin lamina in the x-y plane,

$$I_z = I_x + I_y$$

$$= \underbrace{\sum m_i(x_i^2+y_i^2)}_{I_z} = \underbrace{\sum m_i x_i^2}_{I_x} + \underbrace{\sum m_i y_i^2}_{I_y}$$

since $z = 0$ for the lamina.

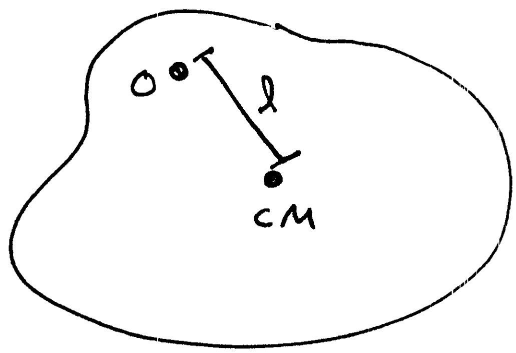
$\Rightarrow$ The moment of inertia about an axis $\perp$ to the lamina is equal to the sum of the moments about two mutually $\perp$ axes through the axis.

Parallel Axis Thm

The moment of inertia about an axis displaced a distance l from a parallel axis through the center of mass is

$$I_o = I_{cm} + m l^2$$

where m is the total mass of the object



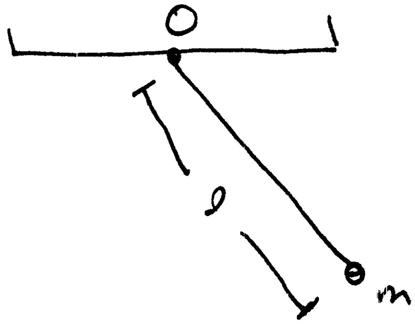
Radius of Gyration (k)

– Distance a point mass must be placed from the axis of rotation to have the same moment of inertia as the body.

$$I_{point} = m k^2$$

$$k = \sqrt{\frac{I}{m}}$$

Simple Pendulum

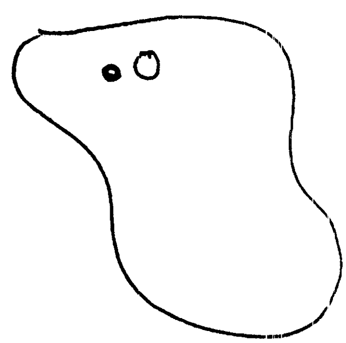


Point mass that rotates about O. The moment of inertia is $I = m l^2$. The frequency of small oscillations is

$$\omega = \sqrt{\frac{mgl}{I}} = \sqrt{g/l}$$

Physical Pendulum

- Rigid body which rotates about fixed axis O.



<u>Body Forces</u> $\sum \vec{F}_i = M\vec{a}_{cm} = \sum m_i a_i \hat{z}$

$$= \sum -m_i g \hat{z} = -Mg\hat{z}$$

<u>Torques due to body forces</u>

$$\vec{N} = \sum \vec{r}_i \times \vec{F}_i = \sum \vec{r}_i \times (-m_i g \hat{z})$$

$$= \left(\sum_i m_i \vec{r}_i\right) \times (-g\hat{z})$$

$$= M\vec{r}_{cm} \times (-g\hat{z}) = \vec{r}_{cm} \times (-Mg\hat{z})$$

$\Longrightarrow$ Body forces exert torque as if all mass at center of mass.

$$|\vec{N}| = |\vec{r}_{cm} \times Mg\hat{z}| = Mgl\sin\theta$$

if axis of rotation chosen as origin.

$$N = I\ddot{\theta} = -Mgl\sin\theta$$

$$\ddot{\theta} + \frac{Mgl}{I}\sin\theta = 0$$

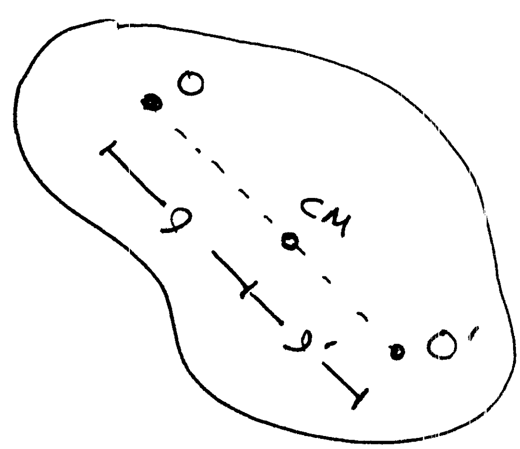
For small oscillations,

$$\ddot{\theta} + \frac{Mgl}{I}\theta = 0$$

Frequency of physical pendulum

$$\omega = \sqrt{\frac{Mgl}{I}}$$

Center of Oscillation (O′) A second point other than O, in line with CM, where oscillation will have same frequency for an axis through O or an axis through O′.



Let $I_0 = m k_0^2$, $I_{cm} = m k_{cm}^2$, where k is radius of gyration.

Parallel axis thm

$$I_o = I_{cm} + m\ell^2$$

$$m k_o^2 = m k_{cm}^2 + m\ell^2$$

Period of Oscillation

$$T_o = 2\pi\sqrt{\frac{I}{mg\ell}}$$

$$= 2\pi\sqrt{\frac{k_{cm}^2 + \ell^2}{g\ell}}$$

Likewise to oscillate about O'

$$T_{o'} = 2\pi\sqrt{\frac{k_{cm}^2 + \ell'^2}{g\ell'}}$$

For periods to be equal, $T_o = T_{o'}$

$$\frac{k_{cm}^2 + \ell'^2}{g\ell'} = \frac{k_{cm}^2 + \ell^2}{g\ell}$$

$$\ell'(k_{cm}^2 + \ell^2) = \ell(k_{cm}^2 + \ell'^2)$$

$$k_{cm}^2(\ell' - \ell) = \ell\ell'^2 - \ell'\ell^2 = \ell\ell'(\ell' - \ell)$$

$$k_{cm}^2 = \ell\ell' \implies \ell' \text{ locates } O'$$

Angular Impulse

$$\Delta L = \Delta p \, l = I_{cm} \Delta \omega = I_{cm} \omega$$

$$\omega = \frac{\Delta p \, l}{I_{cm}}$$

Solve

$$v_0 = \frac{\Delta p}{M} - \frac{\Delta p \, l l'}{I_{cm}}$$

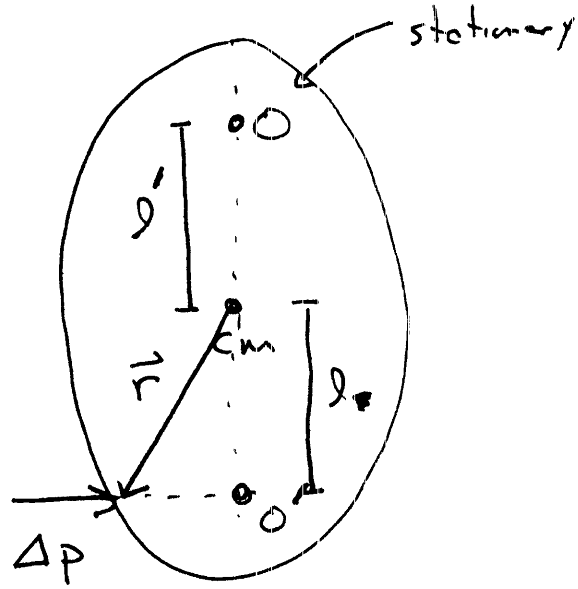
$\Rightarrow$ If $v_0 = 0$,

$$\frac{I_{cm}}{M} = l l' = k_{cm}^2$$

$\Rightarrow$ Center of ~~per~~ percussion for point O' is the same point as the center of oscillation.

$\Rightarrow$ This is where you want to hold the bat.

Center of Percussion - If a body is struck $\perp$ to a line through CM, the center of percussion is the instantaneous axis of rotation, the point with zero velocity.



Separate motion into motion about CM and the motion of the CM.

The velocity of the point O is

$$v_o = v_{cm} - l'\omega$$

Linear impulse $\quad \Delta p = M \Delta v_{cm} = M v_{cm}$