Torque

Recall we could separate the angular momentum about a fixed axis as

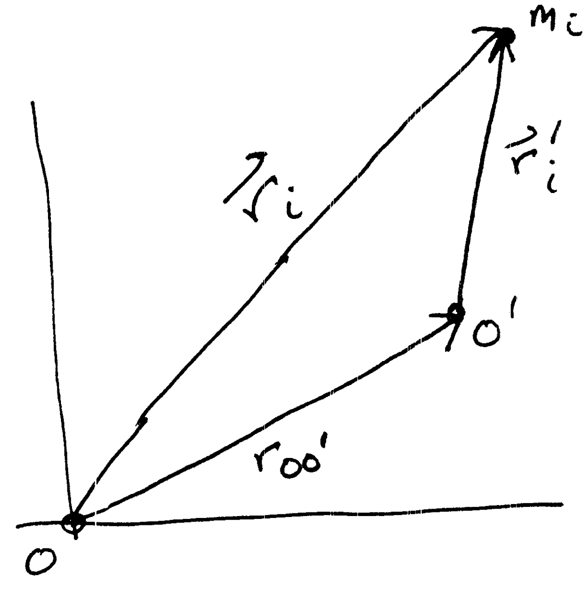
$$\vec{L} = \vec{r}_{cm} \times M\vec{v}_{cm} + \sum m_i \vec{r}_i{}' \times \vec{v}_i{}'$$

where $\vec{r}_i{}'$, $\vec{v}_i{}'$ are relative to CM.

The torque about the fixed axis is given by

$$\vec{N} = \sum \vec{r}_i \times \vec{F}_i = \frac{d\vec{L}}{dt}$$

We would now like to compute motions where the axis of rotation is moving. Let O be the origin of the fixed system and O' the origin of the moving system.



The O and O' frames are connected by

$$\vec{r}_i = \vec{r}_{OO'} + \vec{r}_i\,'$$

$$\vec{v}_i = \dot{\vec{r}}_{OO'} + \dot{\vec{r}}_i\,' = \vec{v}_{OO'} + \vec{v}_i\,'$$

Calculate torque about moving axis, which may be meaningless.

$$\vec{N}\,' \equiv \sum_i \vec{r}_i\,' \times \vec{F}_i$$

where $\vec{F}_i = m_i \dot{\vec{v}}_i = m_i\left(\dot{\vec{v}}_{OO'} + \dot{\vec{v}}_i\,'\right)$

$$\vec{N}\,' = \sum_i \vec{r}_i\,' \times m_i\left(\dot{\vec{v}}_{OO'} + \dot{\vec{v}}_i\,'\right)$$

$$= \left(\sum_i \vec{r}_i\,' \times m_i \dot{\vec{v}}_{OO'}\right) + \sum_i m_i \vec{r}_i\,' \times \dot{\vec{v}}_i\,'$$

$$\frac{d}{dt}\vec{L}\,' = \frac{d}{dt}\sum_i m_i \vec{r}_i\,' \times \vec{v}_i\,'$$

$$= \underbrace{\sum_i m_i \vec{v}_i\,' \times \vec{v}_i\,'}_{0} + \sum m_i \vec{r}_i\,' \times \dot{\vec{v}}_i\,'$$

where I have defined $\vec{L}' = \sum_i m_i \vec{r}_i' \times \vec{v}_i'$ the angular momentum about O'.

$$\vec{N}' = \left(\sum m_i \vec{r}_i'\right) \times \dot{\vec{V}}_{OO'} + \frac{d}{dt}\vec{L}'$$

We would like the first term to vanish, which it does if:

(1) O' is not accelerating, $\dot{\vec{V}}_{OO'} = 0$.

(2) O' is the CM, $\sum m_i \vec{r}_i' = 0$.

(3) $\sum m_i \vec{r}_i' \parallel \dot{\vec{V}}_{OO}$, this is the case if O' is the contact point for a rolling object.

Friction

Static Friction - Object not sliding

$$|F_s| < \mu_s F_N$$

F_N normal force

⟹ To find μ_s, tip incline until slipping begins $\mu_s = \tan\theta$

⟹ Point of application of force does not move so no work.

⟹ Magnitude of force unknown.

Dynamic Friction (Kinetic Friction)

$$|F_K| = \mu_K F_N$$

⟹ Magnitude of force known

⟹ Point of application of force moves, so force does work.

Relation of μ_s and μ_k $\qquad \mu_k < \mu_s$

$\Rightarrow$ If $\mu_k > \mu_s$, slipping would stop immediately after it started.

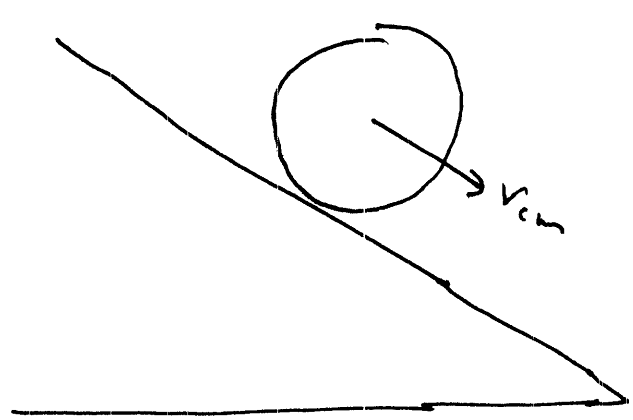
$\Rightarrow$ It is harder to get something moving than to keep it moving.

Condition of Rolling

$$v_{cm} = R\omega$$

$\Rightarrow$ Contact point instantaneously at rest, friction does no work.

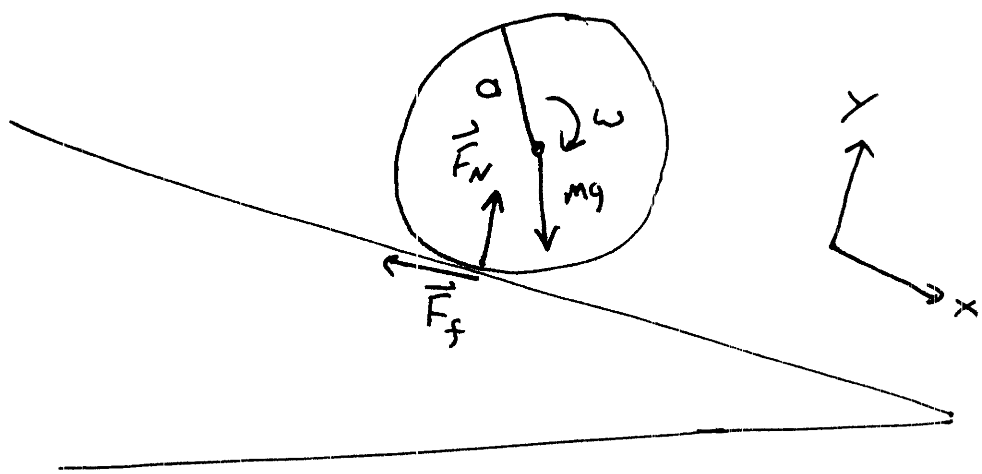
Consider an object rolling down a plane



The object is accelerating, so the frictional force $\vec{F}_f$ is exerting a torque about the center of mass. If this force exceeds $\mu_s F_N$, the object will begin slipping, but we don't know $\vec{F}_f$.

In general, we work out the motion assuming no slipping and check the condition $F_f < \mu_s F_N$.

Work on the ball rolling down the plane.



<u>EOM</u>

$$\vec{F}_{ext} = M\vec{a}_{cm} = \vec{F}_g + \vec{F}_N + \vec{F}_f$$

$$|\vec{N}'| = \left|\frac{d\vec{L}'}{dt}\right| = I_{cm}\dot{\omega} = |\vec{r} \times \vec{F}_f| = |F_f|\,a$$

Note, signs correct given ω definition

Compenent EOM

$$M\ddot{x}_{cm} = Mg\sin\theta - F_f$$

$$M\ddot{y}_{cm} = 0 = -Mg\cos\theta + F_N$$

$$I_{cm}\dot{\omega} = F_f\, a$$

Assume no slipping

$$v_{cm} = a\omega$$

$$a_{cm} = \ddot{x}_{cm} = a\dot{\omega}$$

Solve

$$F_f = \frac{I_{cm}\dot{\omega}}{a} = \frac{I_{cm}\, a_{cm}}{a^2} \qquad \underline{\omega \text{ EOM}}$$

x EOM

$$M a_{cm} = Mg\sin\theta - \frac{I_{cm}\, a_{cm}}{a^2}$$

$$\left(M + \frac{I_{cm}}{a^2}\right) a_{cm} = Mg\sin\theta$$

$$a_{cm} = \ddot{x}_{cm} = \text{constant} = \frac{Mg\sin\theta}{M + \frac{I_{cm}}{a^2}}$$

$$= \frac{g\sin\theta}{1 + \frac{I_{cm}}{Ma^2}}$$

<u>Now test for slipping</u>

$$F_f = \frac{I_{cm}\,a_{cm}}{a^2} = \frac{I_{cm}}{a^2}\left(\frac{g\sin\theta}{1+\frac{I_{cm}}{Ma^2}}\right)$$

$$= \frac{Mg\sin\theta}{1 + \frac{Ma^2}{I_{cm}}} < \mu_s \underbrace{Mg\cos\theta}_{F_N}$$

Slipping occurs when the two terms are equal, call this angle θ_s.

$$\mu_s = \frac{\tan\theta_s}{1 + \frac{Ma^2}{I_{cm}}}$$

~~Slipping stops when condition of rolling met.~~

Two cases

$\theta < \theta_s$ - No slipping

$$a_{cm} = \frac{Mg\sin\theta}{M + \frac{I_{cm}}{a^2}} = \text{constant}$$

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_{cm} t^2$$
$$\theta(t) = \theta_0 + \left(\frac{v_0}{a}\right) t + \frac{1}{2}\frac{a_{cm}}{a} t^2$$

$\theta > \theta_s$ - Slipping

$$F_f = \mu_k Mg\cos\theta \qquad (\text{Friction known})$$

$$M\ddot{x}_{cm} = Mg\sin\theta - F_f$$
$$= Mg\sin\theta - \mu_k Mg\cos\theta$$

$$\ddot{x}_{cm} = g(\sin\theta - \mu_k\cos\theta) = \text{constant}$$

$$x(t) = x_0 + v_0 t + \frac{1}{2} a_{cm} t^2$$

but $\theta(t)$ not known

$$a F_f = I_{cm}\dot{\omega}$$

$$\dot{\omega} = \frac{a F_f}{I_{cm}} = \text{constant}$$

$$\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\dot{\omega} t^2$$

Note, torque tends to increase ω, so condition of rolling will not be met.

$\Rightarrow$ If ball not slipping, energy is conserved

$$E = \text{constant} = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I \omega^2 + mgh$$

$$h = -x \sin\theta$$

$$\omega = \frac{v_{cm}}{a}$$