Three Dimensional Rotations

Inertia Tensor

$$\overleftrightarrow{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$I_{xx} = \sum m_i (y_i^2 + z_i^2)$$ moment about x axis

$$I_{yy} = \sum m_i (x_i^2 + z_i^2)$$ y axis

$$I_{zz} = \sum m_i (x_i^2 + y_i^2)$$ z axis

Products of Inertia

$$I_{xy} = -\sum m_i x_i y_i = I_{yx}$$

etc.

If the instantaneous axis of rotation is

$$\vec{\omega} = \omega \hat{n}$$

the moment about $\vec{\omega}$ is

$$I = \hat{n}^T \overleftrightarrow{I} \hat{n}$$

Ignoring the derivation,

$$\vec{L} = \overleftrightarrow{I}\,\vec{\omega}$$

$\Longrightarrow$ Since $\overleftrightarrow{I}$ is a matrix, $\vec{L}$ and $\vec{\omega}$ do not necessarily point in the same direction.

Kinetic Energy

$$T_{rot} = \frac{1}{2}\,\vec{\omega}^T\,\overleftrightarrow{I}\,\vec{\omega}$$

Principle Axes - There exist a coordinate system where I is diagonal. The axes of this coordinate system are the principle axes of the body. In this coordinate system,

$$\overleftrightarrow{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

and I_1, I_2, I_3 are principle moments.

In this coordinate system,

$$\vec{L} = I_1 \omega_1 \hat{x} + I_2 \omega_2 \hat{y} + I_3 \omega_3 \hat{z}$$

$$T_{rot} = \frac{1}{2}\left(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2\right)$$

Euler's Equations

$$\vec{N} = \frac{d\vec{L}}{dt}$$

Unbelievably useful relation from Chap 5

$$\left.\frac{d\vec{A}}{dt}\right|_{fixed} = \left.\frac{d\vec{A}}{dt}\right|_{rot} + \vec{\omega} \times \vec{A}$$

$$\vec{N} = \left.\frac{dL}{dt}\right|_{fixed} = \underbrace{\left.\frac{dL}{dt}\right|_{rot}}_{\dot{\vec{L}} = \overleftrightarrow{I}\,\dot{\vec{\omega}}} + \underbrace{\vec{\omega} \times \vec{L}}_{\vec{\omega} \times (\overleftrightarrow{I} \cdot \vec{\omega})}$$

Wee!

typo in book

Work everything out - Principle axes - Body fixed coordinate system

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} I_1 \dot{\omega}_1 \\ I_2 \dot{\omega}_2 \\ I_3 \dot{\omega}_3 \end{pmatrix} + \begin{pmatrix} \omega_2 \omega_3 (I_3 - I_2) \\ \omega_3 \omega_1 (I_1 - I_3) \\ \omega_1 \omega_2 (I_2 - I_1) \end{pmatrix}$$

Euler Equations

Solution for free rotation of symmetric object. Let axis 3 be symmetry axis

$$I_s = I_3$$

$$I_1 = I_2 \equiv I$$

$$0 = I\dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I) = 0$$

$$0 = I\dot{\omega}_2 + \omega_3 \omega_1 (I - I_3) = 0$$

$$I_3 \dot{\omega}_3 = 0 \Rightarrow \omega_3 = \text{constant}$$

Define $\Omega = \omega_3 \dfrac{(I_3 - I)}{I}$

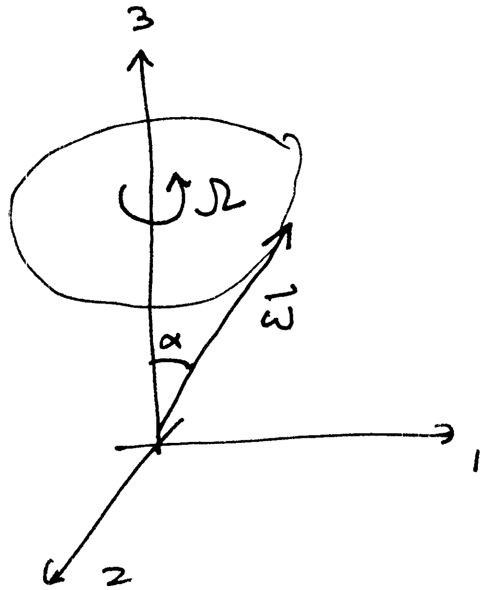
$$\dot{\omega}_1 + \Omega \omega_2 = 0$$

$$\dot{\omega}_2 - \Omega \omega_1 = 0$$

or $\ddot{\omega}_1 + \Omega^2 \omega_1 = 0$

$$\omega_1 = \omega_0 \cos \Omega t$$

$$\omega_2 = \omega_0 \sin \Omega t$$



Relate to fixed coordinate system

⟹ Rotations about coordinate axis are not independant, do not commute.

⟹ Rotations through Euler angles do.

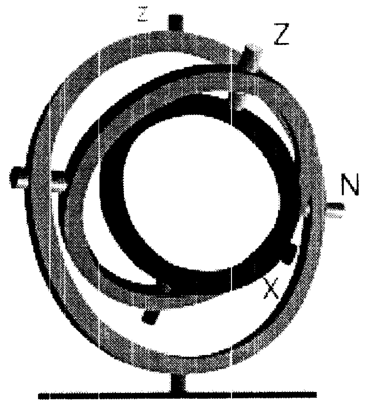
Angle ϕ - Rotate body about z axis

Angle θ - Rotate body about new x axis

Angle ψ - Rotate body about new z axis.

For a symmetric top, $(I_1 = I_2)$

$$T_{rot} = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2$$



Double Gimbel Mount