Central Forces

Central Force - $\vec{F} = f(r)\,\hat{r}$ - Force always directed toward or away from some center, chosen to be the origin.

What is conserved for a central force?

Energy (Yes) -

$$\vec{F} = \underset{f(r)}{\underset{\shortparallel}{F_r}}\,\hat{r} + \underset{0}{\underset{\shortparallel}{F_\theta}}\,\hat{\theta} + \underset{0}{\underset{\shortparallel}{F_\phi}}\,\hat{\phi}$$

Using Griffith's formula for the curl in spherical coordinates.

$$\nabla \times \vec{F} = \frac{1}{r}\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \theta}\,\hat{\theta} - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\,\hat{\phi}$$

keeping only terms with F_r.

so $\nabla \times \vec{F} = 0 \implies$ Force is conservative

$$\vec{F} = -\nabla V$$

Since f depends only on r,

$$f(r) = -\frac{dV}{dr}$$

$$V = -\int f(r)\,dr$$

Ex The potential energy of the gravitational force, $f(r) = \frac{-mMG}{r^2}$

$$V(r) = -\int \frac{-mMG}{r^2}\,dr$$

$$= \frac{-mMG}{r} + C$$

$V(\infty) = 0$

$\implies C = 0$

$$V(r) = \frac{-mMG}{r}$$

Momentum not conserved $\Longrightarrow$ Object exerting central force not in system. Momentum exchanged with external object.

<u>Dfn Angular Momentum</u> ($\vec{L}$) - The angular momentum about the origin is

$$\vec{L} = \vec{r} \times \vec{p}$$

For a central force,

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$= \vec{v} \times \vec{p} + \vec{r} \times \vec{F}$$

But $\quad \vec{v} \times \vec{p} = \vec{v} \times m\vec{v} = m\,\vec{v} \times \vec{v} = 0$

and $\quad \vec{r} \times \vec{F} = \vec{r} \times f(r)\,\hat{r} = r\,f(r)\,\hat{r} \times \hat{r} = 0$

so $\quad \dfrac{d\vec{L}}{dt} = 0 \Longrightarrow$ Angular momentum is conserved.

Since $\vec{L}$ is constant, and $\vec{L} = \vec{r} \times m\vec{v}$, the motion occurs in the plane defined by $\vec{r}$ and $\vec{v}$. The plane is $\perp$ to $\vec{L}$.

$\Rightarrow$ Motion is two-dimensional, let motion occur in x-y plane with $\vec{L}$ in $\hat{z}$ direction.

The text defines $\quad \ell = \dfrac{|\vec{L}|}{m} \quad$ which is also a constant.

Work out $\vec{L}$ in plane-polar coordinates. From kinematics sheet.

$$\vec{r} = r\hat{r}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\vec{L} = \vec{r} \times \vec{p} = m\,\vec{r} \times \vec{v} = m\begin{vmatrix} \hat{r} & \hat{\theta} & \hat{z} \\ r & 0 & 0 \\ \dot{r} & r\dot{\theta} & 0 \end{vmatrix}$$

$$= mr^2\dot{\theta}\,\hat{z}$$

$$\frac{|\vec{L}|}{m} = r^2\dot{\theta} \equiv \ell \quad \text{constant}$$

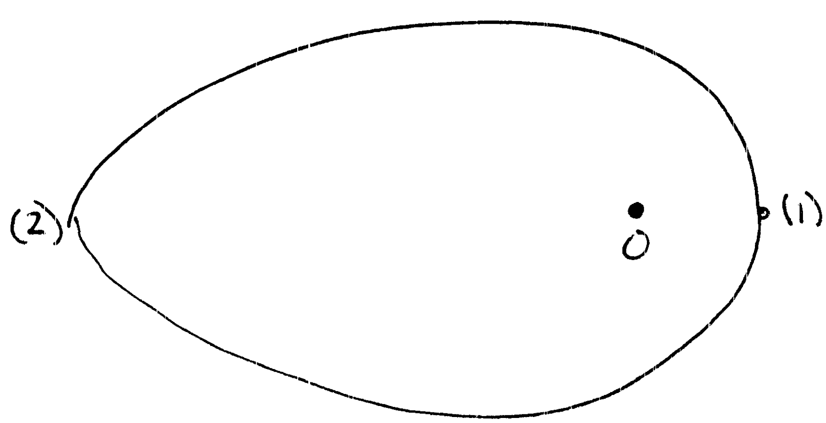
What do we want to know?

I Trajectory $r(t)$, $\theta(t)$

II Orbit - Shape of the trajectory $r(\theta)$.

III Extreme points - Points where $\dot{r} = 0$ (no moving to center) or $\dot{\theta} = 0$, but since $\ell = r^2\dot{\theta}$ there are none of these.

For a gravitational central force, the orbit is an ellipse



At point (1) and (2) $\dot{r}=0$, and $\vec{r} \perp \vec{v}$

so $\quad L = mrv = m\ell \quad$ and $\quad rv = \ell$.

So what can we do with this?

Ex Suppose a particle in a central force has a spiral orbit given by $r=\theta$. Find the trajectory.

$$\ell = r^2\dot{\theta} = \text{constant}$$

$$= \theta^2\dot{\theta} = \theta^2 \frac{d\theta}{dt}$$

$$\int_0^t dt\,\ell = \int_0^\theta \theta^2 d\theta \qquad \text{letting } \theta=0 \text{ at } t=0.$$

$$\ell t = \frac{\theta^3}{3}$$

$$\theta(t) = (3\ell t)^{1/3}$$

$$r(t) = \theta(t) = (3\ell t)^{1/3}$$

Conservation of Energy

$$E = \text{constant} = \frac{1}{2} m v^2 + V(r)$$

$$v^2 = \vec{v} \cdot \vec{v} = \dot{r}^2 + r^2 \dot{\theta}^2$$

$$\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + V(r) = E$$

Because it works, try $\quad u \equiv \frac{1}{r}$.

$$r = \frac{1}{u}$$

$$\dot{r} = -\frac{1}{u^2} \dot{u} = -\frac{1}{u^2} \frac{du}{dt}$$

$$= -\frac{1}{u^2} \frac{du}{d\theta} \frac{d\theta}{dt} = -\frac{\dot{\theta}}{u^2} \frac{du}{d\theta}$$

So

$$E = \frac{1}{2} m \frac{\dot{\theta}^2}{u^4} \left(\frac{du}{d\theta} \right)^2 + \frac{1}{2} m \frac{\dot{\theta}^2}{u^2} + V(u^{-1})$$

Since $\vec{L}$ conserved, $\quad r^2 \dot{\theta} = l$

$$\dot{\theta} = l u^2$$

Energy Equation of Orbit

$$E = \frac{1}{2} m \ell^2 \left(\left(\frac{du}{d\theta} \right)^2 + u^2 \right) + V(u^{-1})$$

Ex What central force would produce spiral trajectory $r = a\,\theta$ (a constant)?

$$\frac{1}{u} = a\,\theta$$

$$u = \frac{1}{a}\,\frac{1}{\theta}$$

$$\frac{du}{d\theta} = -\frac{1}{a}\,\frac{1}{\theta^2} = -a\,\frac{1}{a^2\theta^2} = -a u^2$$

$$E = \frac{1}{2} m \ell^2 \left(a^2 u^4 + u^2 \right) + V(u^{-1})$$

$$V(r) = E - \frac{1}{2} m \ell^2 \left(\frac{a^2}{r^4} + \frac{1}{r^2} \right)$$

$$f(r) = -\frac{dV}{dr} = \frac{1}{2} m \ell^2 \left(-\frac{4a^2}{r^5} - \frac{2}{r^3} \right)$$

$$= -\frac{1}{2} \frac{m \ell^2}{r^3} \left(2 + \frac{4a^2}{r^2} \right)$$

Orbital Equations of Motion

EOM

$$\vec{F} = m\vec{a} = f(r)\,\hat{r}$$

$$= m\left[\ddot{r} - r\dot{\theta}^2\right]\hat{r} + m\left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\hat{\theta}$$

$$\underbrace{\hphantom{m\left[\ddot{r} - r\dot{\theta}^2\right]\hat{r} + m\left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]}}_{\text{Kinematics sheet}}$$

Component EOM

$\hat{r}$: $\quad f(r) = m\ddot{r} - rr\dot{\theta}^2$

$\hat{\theta}$: $\quad 0 = 2m\dot{r}\dot{\theta} + r\ddot{\theta}$

Use conservation of momentum

$$l = r^2\dot{\theta} = \text{constant}$$

$$\frac{dl}{dT} = r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}$$

$$= r\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)$$

So if angular momentum is conserved, $\hat{\theta}$ equation is taken care of.

Solve $\quad f(r) = m\ddot{r} - r\dot{\theta}^2$

Again try $\quad r = \frac{1}{u}$

$$\dot{r} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$= -\frac{\dot{\theta}}{u^2}\frac{du}{d\theta}$$

but $\quad l = r^2\dot{\theta} = \frac{\dot{\theta}}{u^2}$

$$\dot{r} = -l\frac{du}{d\theta}$$

$$\ddot{r} = -l\frac{d}{dt}\frac{du}{d\theta} = -l\frac{d}{d\theta}\frac{du}{dt}$$

$$= -l\left(\frac{d}{d\theta}\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -l\dot{\theta}\frac{d^2u}{d\theta^2} = -l^2u^2\frac{d^2u}{d\theta^2}$$

$$f(u^{-1}) = m\left[-l^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}(l^2u^4)\right]$$

$$= -ml^2u^2\left[\frac{d^2u}{d\theta^2} + u\right]$$

<u>Differential Equation of Orbit</u>

$$\frac{d^2u}{d\theta^2} + u = \frac{f(u^{-1})}{-ml^2u^2}$$

<u>Ex</u> Same problem again, suppose $r = a\theta$

$$\frac{1}{u} = a\theta \qquad u = \frac{1}{a}\frac{1}{\theta}$$

$$\frac{du}{d\theta} = -\frac{1}{a\theta^2} \qquad \frac{d^2u}{d\theta^2} = \frac{2}{a\theta^3}$$

$$= -au^2 \qquad = 2a^2u^3$$

$$\frac{-f(u^{-1})}{ml^2u^2} = \frac{d^2u}{d\theta^2} + u = 2a^2u^3 + u$$

$$f(r) = -\frac{ml^2}{r^2}\left(\frac{2a^2}{r^3} + \frac{1}{r}\right)$$

which is what we got before.