Collisions

Collisions are short range interactions where the details of the forces are unknown. We observe the momenta $\vec{p}_i^{\,I}$ of the initial state and the momenta $\vec{p}_i^{\,F}$ of the final state.

Momentum Conservation – Since no external forces momentum is conserved.

$$\underbrace{\sum_i \vec{p}_i^{\,I}}_{\text{initial}} = \underbrace{\sum_i \vec{p}_i^{\,F}}_{\text{final}}$$

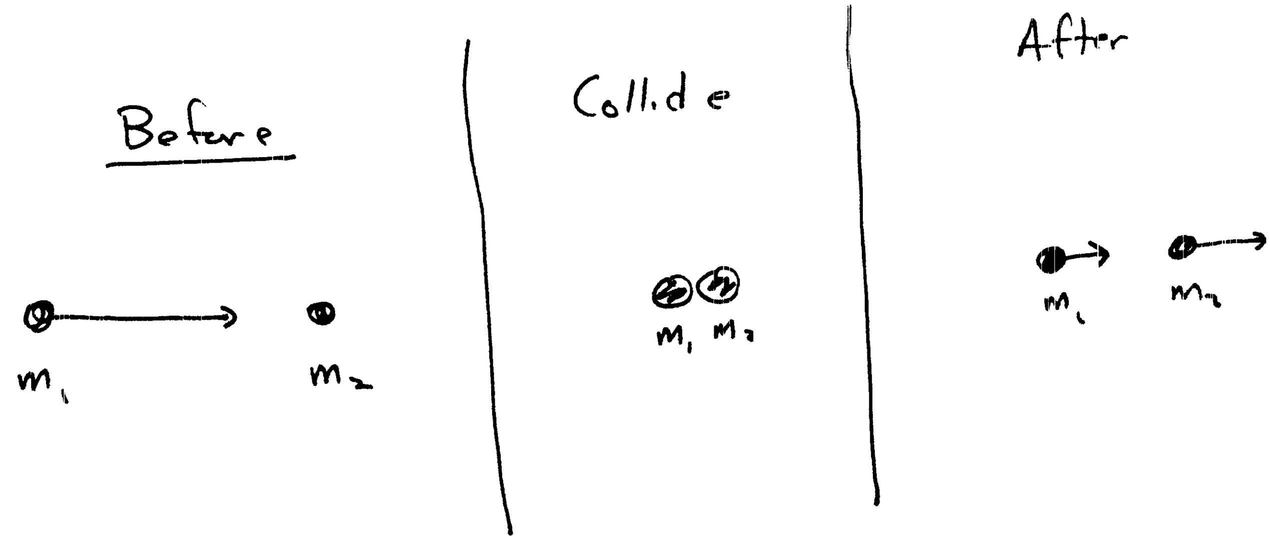
Energy Conservation

$$\sum_i \frac{(\vec{p}_i^{\,I})^2}{2m_i} = \sum_i \frac{(\vec{p}_i^{\,F})^2}{2m_i} + Q$$

Q = Energy lost to or gained from environment

$Q > 0$ if energy lost to environment

Direct Collisions – Particles move on same line before and after collision



Momentum Conservation (2 particles)

$$P_1^I + P_2^I = P_1^F + P_2^F$$

Note: P have signs, $P>0$ represents particle moving toward right

Energy Conservation

$$\frac{(P_1^I)^2}{2m_1} + \frac{(P_2^I)^2}{2m_2} = \frac{(P_1^F)^2}{2m_1} + \frac{(P_2^F)^2}{2m_2} + Q$$

Relative Velocities

Before Collision $\quad v^I \equiv |v_2^I - v_1^I|$

After $\quad v^F \equiv |v_2^F - v_1^F|$

Coefficient of Restitution – How much relative velocity is restored to the particle after collision?

$$\varepsilon = \frac{v^F}{v^I}$$

Elastic Collision – $Q = 0$ – No energy lost to environment.

$$\varepsilon = 1$$

General Final Velocities

$$v_1^F = \frac{(m_1 - \varepsilon m_2)v_1^I + (m_2 + \varepsilon m_2)v_2^I}{m_1 + m_2}$$

$$v_2^F = \frac{(m_1 + \varepsilon m_1)v_1^I + (m_2 - \varepsilon m_1)v_2^I}{m_1 + m_2}$$

<u>Totally Inelastic Collision</u> The two bodies stick together, $v^F = 0$, $\varepsilon = 0$.

$$V_1^F = V_2^F = \frac{m_1 v_1^I + m_2 V_2^I}{m_1 + m_2} = V_{cm}$$

<u>Elastic Collision</u> $(m_1 = m_2 = m)$ $\varepsilon = 1$

$$V_1^F = V_2^I$$

$$V_2^F = V_1^I$$

The particles exchange velocity.

<u>Collision with stationary object</u>

$$V_2^I = V_2^F = 0$$

$$\varepsilon = \frac{|V_1^F|}{|V_1^I|}$$

The ratio of incoming to outgoing velocities

- IF the object is dropped and the stationary object is the ground, the initial and final heights are related by

$$mgh^I = \frac{1}{2} m v_1^{I^2}$$

$$mgh^F = \frac{1}{2} m v_1^{F^2}$$

$$\frac{h_F}{h_I} = \frac{V_1^{F^2}}{V_1^{I^2}} = \varepsilon^2$$

Energy Lost to Environment

$$Q = \frac{1}{2}\mu (v^{I})^2 (1-\varepsilon^2)$$

μ - reduced mass

How much energy of the collision can we use to do other stuff?

Energy that we use becomes energy lost to the environment, therefore the maximum useful energy is Q_{max} which occurs when the collision is totally inelastic, $\varepsilon = 0$

$$Q_{max} = \frac{1}{2}\mu (v^{I})^2$$

Why can't we use all the kinetic energy in the incoming particles?

Recall

$$T = \frac{1}{2} M v_{cm}^2 + \underbrace{\sum \frac{1}{2} m_i v_i'^2}_{T \text{ about CM}}$$

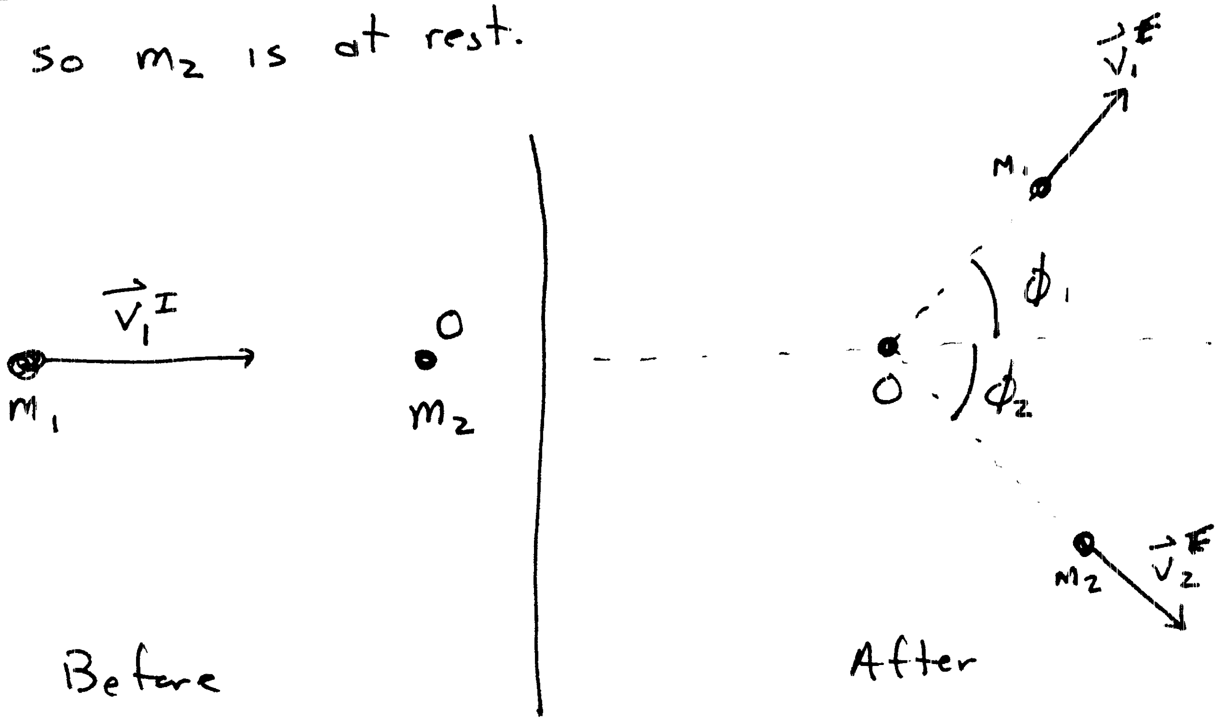
Since $\sum \vec{F}_i = 0$, $\vec{P}_{cm} = \text{constant}$. So we cannot use motion needed to conserve the momentum of the CM.

$$Q_{max} = \frac{1}{2} \sum m_i v_i'^2$$

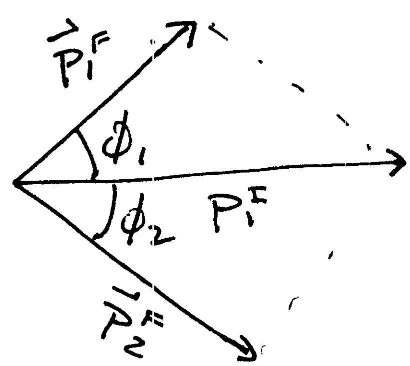
Note, since Q_{max} is relative to CM it does not depend on the coordinate system.

Collisions in Two Dimensions

Lab Coordinate System - Select coordinate system so m_2 is at rest.



Momentum Conservation



$$\vec{P}_1^F = \vec{P}_1^F + \vec{P}_2^F$$

$$p_1^I = p_1^F \cos\phi_1 + p_2^F \cos\phi_2 \qquad x\text{-component}$$

$$0 = p_1^F \sin\phi_1 + p_2^F \sin\phi_2 \qquad y\text{-component}$$

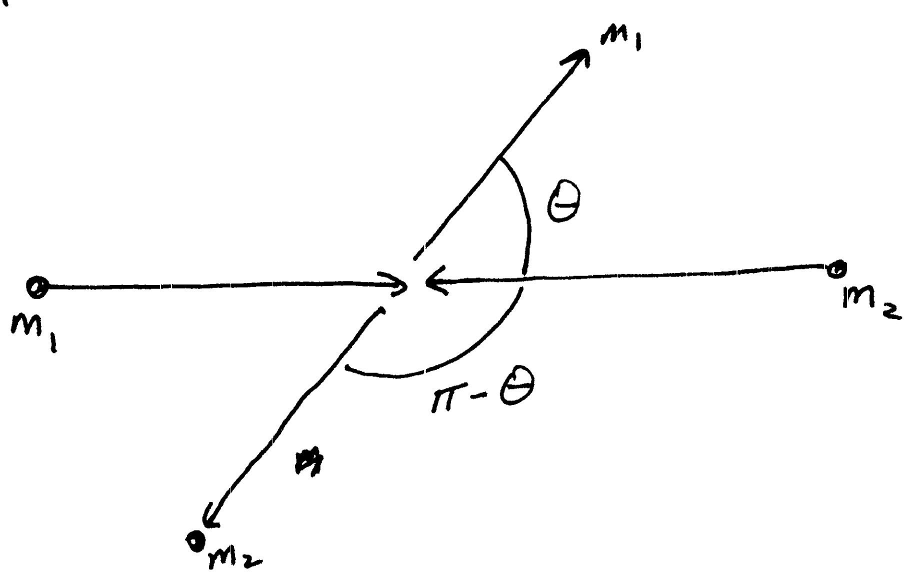
<u>Energy Conservation</u>

$$\frac{(p_1^I)^2}{2m_1} = \frac{(p_1^F)^2}{2m_1} + \frac{(p_2^F)^2}{2m_2} + Q$$

Now consider collision in the Center of Mass coordinate system, where CM is stationary.

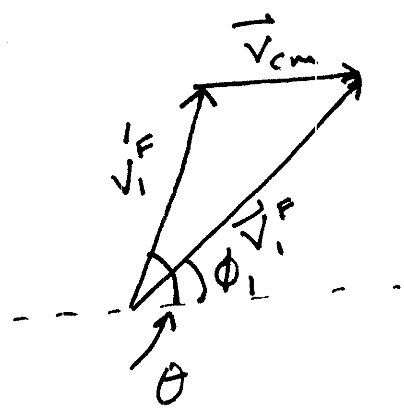
$$\vec{P} = M\vec{V}_{cm} = 0$$

$$\vec{p}_1^{\,I} + \vec{p}_2^{\,I} = 0 = \vec{p}_1^{\,F} + \vec{p}_2^{\,F}$$



Since the total momentum before and after is equal to zero. The initial and final trajectories must be in a line.

Relation between lab and CM coordinates



$$\vec{V} = \vec{V}_{cm} + \vec{V}'$$

$$V_1^F \cos\phi_1 = V_{cm} + V_1'^F \cos\theta \qquad \text{x-component}$$

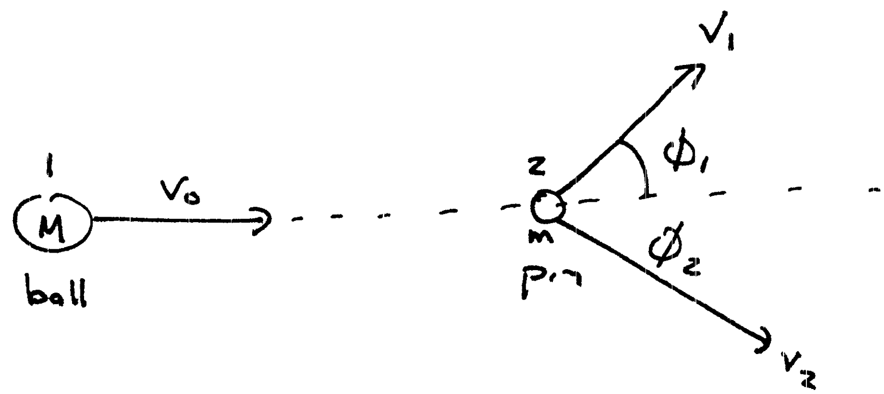
$$V_1^F \sin\phi_1 = V_1'^F \sin\theta \qquad \text{y-component}$$

$$\tan\phi_1 = \frac{V_1'^F \sin\theta}{V_{cm} + V_1'^F \cos\theta} = \frac{\sin\theta}{\gamma + \cos\theta}$$

$$\gamma \equiv \frac{V_{cm}}{V_1'^F}$$

For elastic collisions, $\gamma = m_1/m_2$

Ex Producing a split



Let $M = 5m$, $\phi_2 = 30°$

Momentum

$$\vec{P}_1^{\,I} = \vec{P}_1^{\,F} + \vec{P}_2^{\,F}$$

x-component

$$5mv_0 = 5mv_1^F \cos\phi_1 + mv_2^F \cos 30°$$

y-component

$$0 = 5mv_1^F \sin\phi_1 + mv_2^F \cos 30°$$

Assume collision is elastic and rotational energy can be ignored.

Energy $\frac{1}{2}(5m)v_0^2 = \frac{1}{2}(5m)v_1^{F^2} + \frac{1}{2}mv_2^{F^2}$

$$5v_0^2 = 5v_1^{F^2} + v_2^{F^2}$$

$$v_0^2 - \frac{1}{5}v_2^{F^2} = v_1^{F^2}$$

$$\cancel{v_0 - v_1^F \cos\phi_1}$$

Work on 3 equations

$$v_0 - \frac{1}{5}v_2^F \cos 30 = v_1^F \cos\phi_1 \qquad (1)$$

$$\frac{1}{5}v_2^F \sin 30 = v_1^F \sin\phi_1 \qquad (2)$$

$(1)^2 + (2)^2$

$$v_0^2 - \frac{2}{5}v_0 v_2^F \cos 30 + \frac{1}{25}v_2^{F^2}\cos^2 30 + \frac{1}{25}v_2^{F^2}\sin^2 30$$

$$= v_1^{F^2}\cos^2\phi_1 + v_1^{F^2}\sin^2\phi_1$$

$$v_0^2 - \frac{2}{5}v_0 v_2^F \cos 30 + \frac{1}{25}v_2^{F^2} = v_1^{F^2} \qquad (3)$$

(3) - Energy

$$-\frac{2}{5} v_0 v_2^F \cos 30 + \frac{1}{25} v_2^{F\,2} + \frac{1}{5} v_2^{F\,2} = 0$$

$$-10 v_0 \cos 30^\circ + 6 v_2^F = 0$$

$$\boxed{v_2^F = \frac{10}{6} v_0 \cos 30 = 1.44 v_0}$$

Find scattering angle (2)/(1)

$$\tan\phi_1 = \frac{\frac{1}{5} v_2^F \sin 30}{v_0 - \frac{1}{5} v_2^F \cos 30}$$

$$= 0.19 \qquad \phi_1 \sim 10^\circ$$

Final Velocity of ball

$$v_1^F = \sqrt{v_0^2 - \frac{1}{5} v_2^{F\,2}} = 0.77 v_0$$

$$\vec{v}_2^F = 1.44 v_0 \left(\cos 30, \sin 30, 0\right)$$

$$\vec{v}_1^F = 0.77 v_0 \left(\cos 10, \sin 10, 0\right)$$

Now what if we look at the collision in the CM?

$$\tan \phi_1 = \frac{\sin \theta}{\gamma + \cos\theta}$$

θ scattering angle in CM

Since elastic $\gamma = \frac{m_1}{m_2} = 5$

We found $\phi_1 = 10°$, solution of

$$\tan 10° = \frac{\sin\theta}{5 + \cos\theta}$$

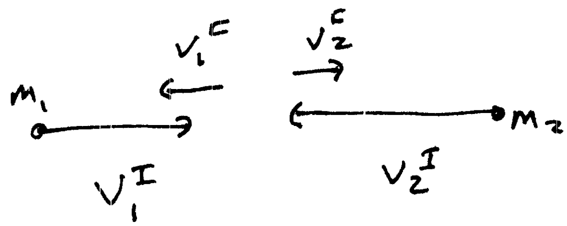
yields $\theta = 70°$

What is $\vec{v}_{cm}$ before collision?

$$|\vec{V}_{cm}| = \frac{|\vec{P}|}{M} = \frac{5 m v_0}{5m + m} = \frac{5}{6} V_0$$

Additional Collision Notes

Elastic Collision



$$m_1 v_1^I + m_2 v_2^I = m_1 v_1^F + m_2 v_2^F \qquad \text{momentum}$$

$$\frac{1}{2} m_1 {v_1^I}^2 + \frac{1}{2} m_2 {v_2^I}^2 = \frac{1}{2} m_1 {v_2^F}^2 + \frac{1}{2} m_2 {v_2^F}^2$$

$$m_1\left({v_1^F}^2 - {v_1^I}^2\right) = m_2\left({v_2^F}^2 - {v_2^I}^2\right)$$

$$m_1\left(v_1^F + v_1^I\right)\left(v_1^F - v_1^I\right) = m_2\left(v_2^F + v_2^F\right)\left(v_2^F - v_2^I\right)$$

Divide by momentum equation

$$v_1^F - v_1^I = v_2^F - v_2^I$$

$$\Rightarrow \quad \varepsilon = 1$$

Totally Inelastic Collision - Two bodies stick together: $v_1^F = v_2^F$, $\varepsilon = 0$, since $v^F = 0$

Momentum

$$m_1 v_1^I + m_2 v_2^I = (m_1 + m_2) v_1^F$$

$$v_1^F = \frac{m_1 v_1^I + m_2 v_2^I}{m_1 + m_2}$$

Energy

$$\frac{1}{2} m_1 v_1^{I^2} + \frac{1}{2} m_2 v_2^{I^2} = \frac{1}{2}(m_1 + m_2) v_1^{F^2} + Q$$

$$= \frac{1}{2(m_1+m_2)} \left(m_1 v_1^I + m_2 v_2^I\right)^2 + Q$$

$$= \frac{1}{2(m_1+m_2)} \left(m_1^2 v_1^{I^2} + 2 m_1 m_2 v_1^I v_2^I + m_2^2 v_2^{I^2}\right)$$

$$\underline{m_1^2 v_1^{I^2}} + m_1 m_2 v_1^{I^2} + m_1 m_2 v_2^{I^2} + \underline{m_2^2 v_2^{I^2}} = \underline{m_1^2 v_1^{I^2}} + 2 m_1 m_2 v_1^I v_2^I + \underline{m_2^2 v_2^{I^2}} + 2(m_1 + m_2) Q$$

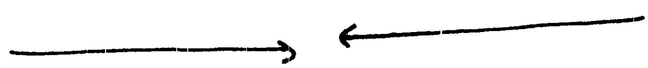
&

$$m_1 m_2 \left(v_1^{I^2} - 2 v_1^I v_2^I + v_2^{I^2}\right) = 2(m_1 + m_2) Q$$

$$Q = \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)\left(v_1^I - v_2^I\right)^2$$

$$= \frac{1}{2}\mu\, v^{I\,2}$$

Lets consider collision in CM



$$V_{cm} = \frac{P_{cm}}{M} = \frac{m_1 v_1^I + m_2 v_2^I}{m_1+m_2} = v_1^F = v_2^F$$

In center of mass, particles stick together and are stationary. All kinetic energy is lost.

$$Q = \underbrace{\frac{1}{2}m_1\left(v_1^{I'}\right)^2 + \frac{1}{2}m_2\left(v_2^{I'}\right)^2}_{\text{Relative to CM}}$$

$$v_1^{I'} + V_{cm} = v_1^I \qquad v_2^{I'} + V_{cm} = v_2^I$$

$$Q = \frac{1}{2}m_1\left(v_1^I - V_{cm}\right)^2 + \frac{1}{2}m_2\left(v_2^I - V_{cm}\right)^2$$

$$Q = \frac{1}{2} m_1 \left(v_1^{I^2} - 2 v_{cm} v_1^I + v_{cm}^2 \right)$$

$$+ \frac{1}{2} m_2 \left(v_2^{I^2} - 2 v_2^I v_{cm} + v_{cm}^2 \right)$$

$$= \frac{1}{2} m_1 v_1^{I^2} + \frac{1}{2} m_2 v_2^{I^2} + \frac{1}{2} (m_1 + m_2) v_{cm}^2$$

$$\underbrace{\underbrace{- m_1 v_{cm} v_1^I - m_2 v_{cm} v_2^I}_{- v_{cm} \left(m_1 v_1^I + m_2 v_2^I \right)}}_{-(m_1 + m_2) v_{cm}^2}$$

$$Q = \underbrace{\frac{1}{2} m_1 v_1^{I^2} + \frac{1}{2} m_2 v_2^{I^2}}_{\text{Total initial } T} - \underbrace{\frac{1}{2} (m_1 + m_2) v_{cm}^2}_{\text{Kinetic energy of center of mass}}$$

Center of Mass

$$\vec{p}_1^{\,I} + \vec{p}_2^{\,I} = 0$$

$$m_1\vec{v}_1^{\,I} + m_2\vec{v}_2^{\,I} = 0$$

$$\vec{v}_2^{\,I} = -\frac{m_1}{m_2}\vec{v}_1^{\,I}$$

Energy

$$\frac{p_1^{I^2}}{2m_1} + \frac{p_2^{I^2}}{2m_2} = \frac{p_1^{F^2}}{2m_1} + \frac{p_2^{F^2}}{2m_2} + Q$$

Since CM

$$\frac{p_1^{I^2}}{2m_1} + \frac{p_1^{I^2}}{2m_2} = \frac{p_1^{F^2}}{2m_1} + \frac{p_1^{F^2}}{2m_2} + Q$$

$$\cancel{\frac{1}{m_1+m_2}} \qquad \frac{1}{m_1} + \frac{1}{m_2} = \frac{m_1+m_2}{m_1 m_2} = \frac{1}{\mu}$$

$$\frac{p_1^{I^2}}{2\mu} = \frac{p_1^{F^2}}{2\mu} + Q$$

Note, velocities on previous page CM velocities.

We can write these in terms of the lab velocities

$$V_1^{LI} = V_{cm} + V_1^{I}$$

$$V_{cm} = \frac{P_{cm}}{m} = \frac{m_1 V_1^{LI} + m_2 V_2^{LI}}{m_1 + m_2}$$

$$V_1^{I} = V_1^{LI} - V_{cm}$$

$$= V_1^{LI} - \left(\frac{m_1 V_1^{LI} + m_2 V_2^{LI}}{m_1 + m_2} \right)$$

$$= \frac{m_2}{m_1 + m_2} V_1^{LI}$$

The two coordinates are connected by

$$\tan \phi_1 = \frac{\sin \theta}{\gamma + \cos \theta}$$

$$\gamma = \frac{V_{cm}}{V_1^{F}}$$

If $Q = 0$,

$$\frac{{P_1^I}^2}{2\mu} = \frac{{P^F}^2}{2m}$$

$$V_1^I = V_1^F$$

$$\gamma = \frac{V_{cm}}{V_1^I} = \frac{\frac{m_1 V_1}{m_1 + m_2}}{\frac{m_2 V_1}{m_1 + m_2}} = \frac{m_1}{m_2}$$