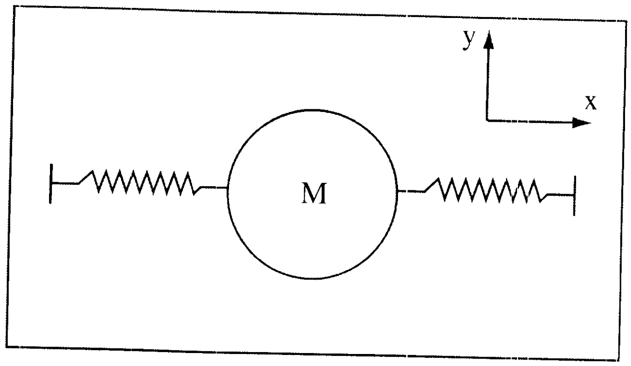
Mechanics Fall 2009 - Final Exam

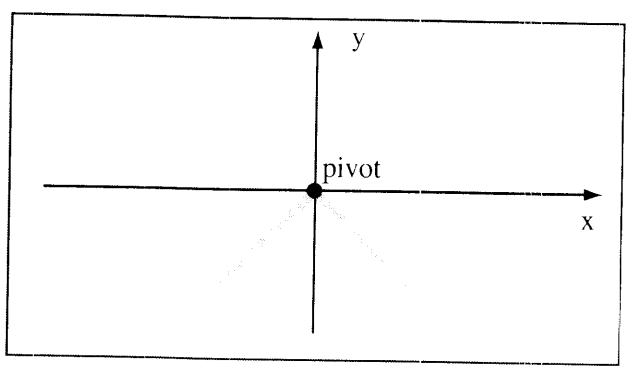
Work four of the five problems. Place the problems in the order you wish them graded. The first two problems form the first half test; the second two problems form the second half test.

Problem 4.1 A disk of mass M and radius R is free to move on a frictionless surface. The disk may rotate and the center may move in the x or y direction. Two springs of spring constant k are attached as shown. The system is in equilibrium in the position drawn. The length of each spring in equilibrium is d. Compute the Lagrangian of the system. You do NOT need to find the equations of motion; they are a mess.



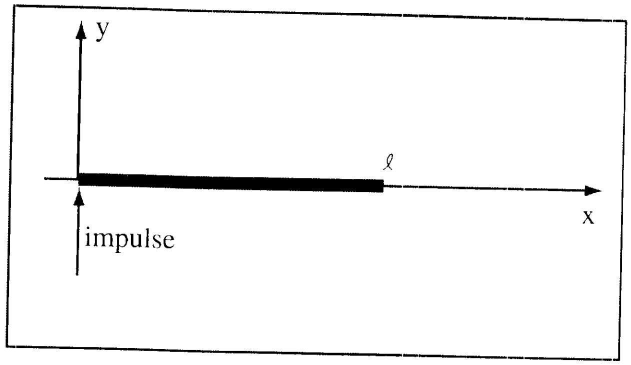
Problem 4.2 Consider a particle of mass m dropped through a hole through the center of the earth. Model the earth as an object of uniform mass density. The radius of the earth is R_e and the mass of the earth is M_e. Find the force of gravity as a function of r (which you did in the homework). If the particle experiences a linear drag force $|cv|$ where v is the velocity, compute the frequency of damped oscillations. Compute the damping constant if the system is to be critically damped, where the particle comes to stop at the center of the earth without oscillation.

Problem 4.3 A physical pendulum is formed of two rods of length ℓ and mass m mounted at right angles as drawn. The pendulum is mounted to pivot about the point where the two rods are joined. The system is shown in equilibrium as shown below. Compute the center of mass in the equilibrium location drawn in the figure. Compute the frequency of small oscillations.

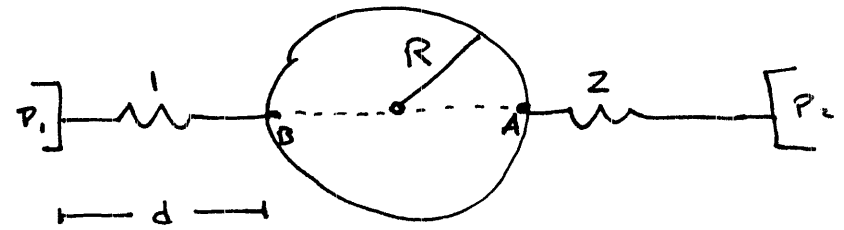


Problem 4.4 Almost all asteroids have semi-major axes from 2A.U. to 4A.U., where 1A.U.= 1.5×10^{11}m, the radius of earth's orbit. This is good, because we don't want asteroids sharing earth's orbit. The eccentricities of the majority of asteroids vary from 0 to 0.4. If an asteroid has semi-major axis 2A.U., what eccentricity must it have to have a perihelion distances of 1A.U. and have a chance of hitting us? At this eccentricity, what is its period? The mass of the sun is $M_\circ = 2 \times 10^{30}$kg and $G = 6.67 \times 10^{-11} Nm^2/kg^2$.

Problem 4.5 A rod has total mass M, length ℓ, and variable linear mass density $\lambda = \gamma x^2$ where γ is a constant that you must determine. The rod lays on a frictionless surface. The rod is struck perpendicular to its axis. How far from the origin must the rod be struck so the instantaneous axis of rotation is its far end $x = \ell$?



4.1



Let the center of the disk be O.

The Location of the center of the disk is (x, y) and the orientation of the dashed line, θ.

The location of point A is

$$x_A = x + R\cos\theta \qquad y_A = y + R\sin\theta$$

and the point B

$$x_B = x - R\cos\theta \qquad y_B = y - R\sin\theta$$

The location of $P_1 = (-(R+d), 0, 0)$ and the location of $P_2 = (R+d, 0, 0)$

The kinetic energy is

$$T = \frac{1}{2} M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I \dot{\theta}^2$$

The potential energy is

$$V = \frac{1}{2} k \Delta r_1^2 + \frac{1}{2} k \Delta r_2^2$$

$$= \frac{1}{2} k \left(\left[x_B - (-R+d) \right]^2 + y_B^2 \right)$$

$$+ \frac{1}{2} k \left(\left[x_A - (R+d) \right]^2 + y_A^2 \right)$$

$$V = \frac{1}{2} k \left(\left[x - R\cos\theta + R + d \right]^2 + (y - R\sin\theta)^2 \right)$$

$$+ \frac{1}{2} k \left(\left[x - R - d + R\cos\theta \right]^2 + (y + R\sin\theta)^2 \right)$$

4.2 The mass density $\rho = \frac{M_e}{\frac{4}{3}\pi R_e^3}$

Using Gauss' law for gravity,

$$4\pi r^2 g = \oint \vec{g} \cdot \hat{n}\, dA = -4\pi G M_{enc}$$

$$\vec{g}(r) = -\frac{G M_{enc}}{r^2}\hat{r}$$

For a spherical surface inside the earth

$$M_{enc} = \frac{4}{3}\pi r^3 \rho$$

$$\vec{g} = -\frac{G M_{enc}}{r^2} = -\frac{G \frac{4}{3}\pi r^3 \rho}{r^2}$$

$$= -\frac{4}{3}\pi \rho G r \hat{r} = -\Gamma r$$

For a particle falling through the earth's center

$$m\ddot{r} = -mg - cv$$

$$= -m\Gamma r - cv$$

$$\ddot{r} + \frac{c}{m}\dot{r} + \Gamma r = 0$$

The damping constant is as usual

$$\gamma = \frac{c}{2m}$$

and the natural frequency $\omega_0^2 = \Gamma = \frac{4}{3}\pi\rho G$

The frequency of damped oscillations (implying underdamping)

is $$\omega_d = \sqrt{\omega_0^2 - \gamma^2}$$

$$= \sqrt{\left(\frac{4}{3}\pi\rho G\right)^2 - \left(\frac{c}{2m}\right)^2}$$

The critical damping constant is given by

$$\gamma = \omega_0$$

$$\frac{c}{2m} = \sqrt{\frac{4}{3}\pi\rho G}$$

$$c = 2m\sqrt{\frac{4}{3}\pi\rho G}$$

Let's try to clean up Γ some

$$|\vec{g}|_{surface} = g = \frac{M_e G}{R_e^2} = 9.81\ m/s^2$$

$$M_e = g R_e^2 / G$$

$$\rho = \frac{M_e}{\frac{4}{3}\pi R_e^3} = \frac{g R_e^2}{\frac{4}{3}\pi G R_e^3}$$

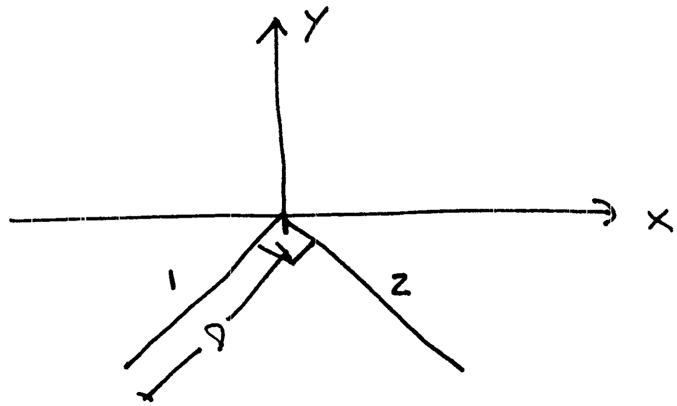
$$= \frac{g}{\frac{4}{3}\pi G R_e}$$

$$\omega_0^2 = \Gamma = \frac{4}{3}\pi\rho G = \left(\frac{4}{3}\pi G\right)\frac{g}{\frac{4}{3}\pi G R_e}$$

$$= \frac{g}{R_e}$$

$$C_{critical} = 2m\sqrt{\frac{g}{R_e}}$$

4.3



The center of mass of the individual rods is at the center of the rods at

$$\vec{r}_1 = \left(-\frac{l}{\sqrt{2}}, \frac{-l}{\sqrt{2}}\right) \quad \text{and} \quad \vec{r}_2 = \left(\frac{l}{\sqrt{2}}, \frac{-l}{\sqrt{2}}\right)$$

The center of mass of the system is found by

$$\vec{r}_{cm} = \frac{1}{2m}\left[m\vec{r}_1 + m\vec{r}_2\right]$$

$$= \frac{1}{2}\left(0, \frac{-2l}{2\sqrt{2}}\right) = \left(0, \frac{-l}{2\sqrt{2}}\right)$$

obviously

The moment of inertia of the system is the sum of the moments

$$I = I_1 + I_2$$

$$= \frac{ml^2}{3} + \frac{ml^2}{3} = \frac{2ml^2}{3}$$

Using the moment of a rod pivoted about its end.

The EOM for the pendulum is

$$N = I\dot{\omega} = -mg\,l_{cm}\sin\theta$$

For small oscillations,

$$I\ddot{\theta} + mg\,l_{cm}\,\theta = 0$$

$$\ddot{\theta} + \frac{mg\,l_{cm}}{I}\theta = 0$$

$$\omega^2 = \frac{mg\,l_{cm}}{I} = \frac{mg\left(l/2\sqrt{2}\right)}{\frac{2ml^2}{3}}$$

$$= \frac{3}{4\sqrt{2}}\frac{g}{l}$$

(4.4) $a = 3 \times 10^{11}\ m$

$r_0 = 1.5 \times 10^{11}\ m$

The relation of perihelion distance r_0 to semi-major axis a.

$$r_0 = \frac{a(1-\varepsilon^2)}{1+\varepsilon} = a(1-\varepsilon)$$

$$\boxed{\varepsilon = 1 - \frac{r_0}{a} = \frac{1}{2}}$$ cool

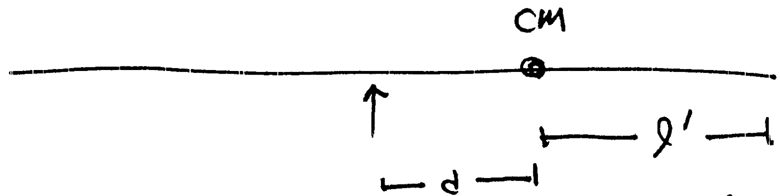
<u>Period</u>

$$T^2 = \frac{4\pi^2}{G M_0} a^3$$

$$= \frac{4\pi^2}{\left(6.67 \times 10^{-11}\ \frac{Nm^2}{kg^2}\right)\left(2 \times 10^{30}\ kg\right)} \cdot \left(3 \times 10^{11}\ m\right)^3$$

$$T = 9 \times 10^7\ s = 2.8 \text{ years}$$ - not bad

Center of Percussion - Instantaneous axis of rotation.



$l' = \frac{1}{4} l$

$$k_{cm}^2 = d\,l'$$

$$d = \frac{k_{cm}^2}{l'} = \frac{\frac{3}{80}\,l^2}{\frac{1}{4}\,l} = \frac{3}{20}\,l$$

4.5

<u>Total mass of rod</u>

$$M = \int_0^{\ell} dm = \int_0^{\ell} \gamma x^2 \, dx$$

$$= \frac{\gamma \ell^3}{3}$$

$$\gamma = \frac{3M}{\ell^3}$$

<u>Center of mass of rod</u>

$$x_{cm} = \frac{1}{M} \int_0^{\ell} x \, dm$$

$$= \frac{1}{M} \int_0^{\ell} \gamma x^3 \, dm$$

$$= \frac{1}{M} \frac{\gamma \ell^4}{4}$$

$$= \frac{\cancel{\gamma}}{4m} \left(\frac{3M}{\ell^3} \right) \ell^4$$

$$= \frac{3}{4} \ell$$

Moment of Inertia (about end)

$$I_{end} = \int_0^{l} r^2 dm$$

$$= \gamma \int_0^{l} x^4 dx$$

$$= \frac{\gamma l^5}{5} = \left(\frac{3M}{l^3}\right)\left(\frac{l^5}{5}\right) = \frac{3}{5} M l^2$$

Moment of Inertia about CM - Parallel-axis thm

$$I_{end} = I_{cm} + m l_{cm}^2$$

$$I_{cm} = I_{end} - m l_{cm}^2$$

$$= \frac{3}{5} m l^2 - \left(m \left(\frac{3}{4}\right)^2 l^2\right)$$

$$= \left(\frac{3}{5} - \frac{9}{16}\right) m l^2$$

$$= \left(\frac{48}{80} - \frac{45}{80}\right) m l^2 = \frac{3}{80} m l^2$$

$$k_{cm}^2 = \frac{3}{80} l^2$$