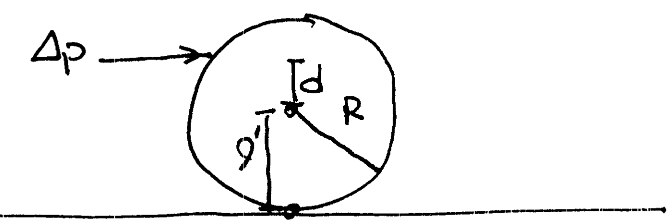
(1) A pool ball of mass M and radius R is struck a distance d above its center. How far above the center must it be struck so the point of contact is the instantaneous axis of rotation? ~~A~~ If struck at this point, is the condition of rolling necessarily met?



Sln For the contact point to be the instantaneous axis of rotation, the ball must be struck at the center of percussion.

$$K_{cm}^2 = \ell \ell' = dR$$

$$d = \frac{K_{cm}^2}{R} =$$

$$K_{cm}^2 = \frac{I_{cm}}{M} = \frac{2}{5}R^2 \quad \text{sphere}$$

$$d = \frac{2}{5}R$$

If struck at this point,

$$V_{cm} = \frac{\Delta p}{M}$$

The rotational velocity is found by conserving angular momentum,

$$I_{cm}\omega = |\vec{r} \times \Delta\vec{p}| = \Delta p\, d$$

$$\omega = \frac{\Delta p\, d}{I_{cm}}$$

$$R\omega = \frac{\Delta p\, d\, R}{I_{cm}}$$

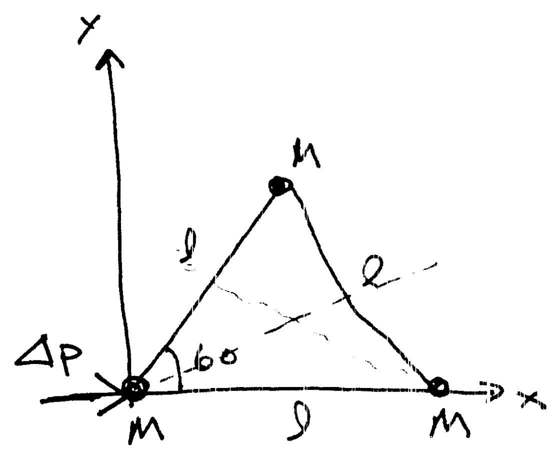
From previous, $K_{cm}^2 = dR$ $\quad I_{cm} = M K_{cm}^2 = MRd$

$$R\omega = \frac{\Delta p\, d\, R}{MRd} = \frac{\Delta p}{M} = V_{cm}$$

$\Longrightarrow$ condition of rolling is satisfied at all Δp.

(2) 3 masses a held in an equilateral triangle by massless rods of length ℓ.

The triangle is struck imparting momentum Δp in the x-direction as shown. Describe the motion



<u>Sln</u>

The location of the masses are $(0,0)$, $(0,\ell)$ and $\left(\ell/2, \ell \sin 60\right) = \frac{\ell}{2}(1, \sqrt{3})$

<u>Center of Mass</u>

$$\vec{r}_{cm} = \frac{1}{3M} \sum m_i \vec{r}_i$$

$$= \frac{1}{3M}\left(M(0,0) + M(\ell,\ell) + \frac{M\ell}{2}(1, \sqrt{3}) \right)$$

$x_{cm} = \ell/2$ as expected.

$\cancel{y_{cm} = \frac{1}{3M}\left(M\ell + \frac{M\ell\sqrt{3}}{2} \right) = \frac{1}{3}\ell\left(1 + \frac{\sqrt{3}}{2}\right)}$ $\cancel{= 0.62\ell}$

$$y_{cm} = \frac{\ell}{2\sqrt{3}}$$

The center of mass moves at a constant velocity

$$\vec{V}_{cm} = \frac{\Delta p}{3M} \hat{x}$$

The object rotates about the CM with constant angular velocity

$$I_{cm} \omega = |\vec{r} \times \Delta p| = r_{cm} \Delta p \sin \theta$$

$$r_{cm} = \sqrt{x_{cm}^2 + y_{cm}^2} = \sqrt{\left(\ell/2\right)^2 + \left(\frac{\ell}{2\sqrt{3}}\right)^2}$$

$$= \frac{\ell}{2} \sqrt{1 + \frac{1}{3}} = \frac{\ell}{\sqrt{3}}$$

$$\sin 30 = \frac{1}{2}$$

$$I_{cm} \omega = \frac{\Delta p \, \ell}{2\sqrt{3}} = \Delta p \, y_{cm}$$

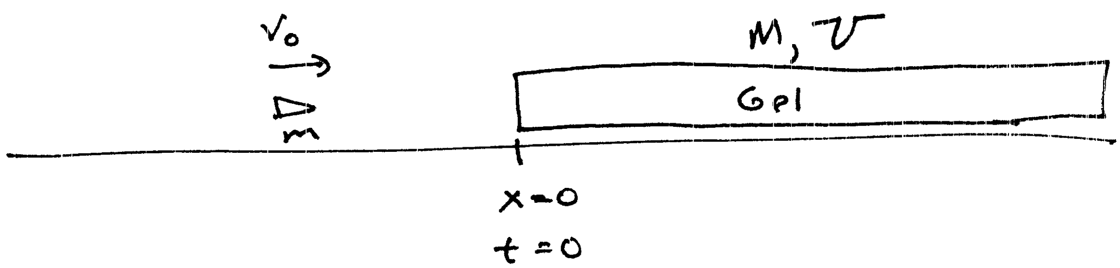
$$I_{cm} = \sum m_i r_i^2 = 3M \left(\frac{\ell}{\sqrt{3}}\right)^2$$

$$= M \ell^2$$

$$\omega = \frac{\Delta p \, \ell}{2\sqrt{3}} = \frac{\Delta p \, \ell}{2\sqrt{3} \, M \ell^2} = \frac{\Delta p}{2\sqrt{3} \, M \ell}$$

③ Bullet is fired into a gel-filled tube that slides on a frictionless surface. The bullet experiences a linear drag force. Report the velocity of the bullet as a function of time and the final velocity of The bullet-tube system.

Sln



Linear Drag Force

$$\vec{F} = -c(v - \mathcal{V}) = m\dot{v}$$

Conserve Momentum

$$mv_0 = mv + M\mathcal{V}$$

$$\mathcal{V} = \frac{m}{M}(v_0 - v)$$

$$m\dot{v} = -c\left(v - \frac{m}{M}(v_0 - v)\right)$$

$$= \frac{cm}{M}v_0 - \left(1 + \frac{m}{M}\right)cv$$

$$\dot{v} = \frac{c}{M}v_0 - \underbrace{\left(\frac{1}{m} + \frac{1}{M}\right)}_{\frac{1}{\mu}}cv$$

<u>Reduced Mass</u> $\quad \mu = \dfrac{mM}{m+M} = \dfrac{1}{\frac{1}{m} + \frac{1}{M}}$

$$\dot{v} = \frac{c}{M}v_0 - \frac{c}{\mu}v = \frac{dv}{dt}$$

$$\int_{v_0}^{v} \frac{dv}{\frac{cv_0}{M} - \frac{c}{\mu}v} = \int_0^t dt$$

$$-\frac{\mu}{c}\ln\left(\frac{cv_0}{M}-\frac{c}{\mu}v\right)\Bigg|_{v_0}^{v} = t$$

$$\ln\left(\frac{cv_0}{M}-\frac{c}{\mu}v\right)-\ln\left(\frac{cv_0}{M}-\frac{cv_0}{\mu}\right) = -\frac{ct}{\mu}$$

$$\frac{1}{M}-\frac{1}{\mu} = \frac{1}{M}-\left(\frac{1}{m}+\frac{1}{M}\right)$$

$$= -\frac{1}{m}$$

$$\ln\left(\frac{cv_0}{M}-\frac{cv_0}{\mu}\right) = \ln\left(-\frac{cv_0}{m}\right)$$

$$\ln\left(\frac{m}{\mu}\frac{v}{v_0}-\frac{m}{M}\right) = -\frac{ct}{\mu}$$

$$\ln\left(\frac{m+M}{M}\frac{v}{v_0}-\frac{m}{M}\right)$$

$$\ln\left(\frac{v}{v_0}+\cancel{\frac{v}{v_0}}\frac{m}{M}\left(\frac{v}{v_0}-1\right)\right) = -\frac{ct}{\mu}$$

$$\frac{v}{v_0}\left(\frac{m+M}{M}\right) - \frac{m}{M} = e^{-ct/\mu}$$

$$\frac{v}{v_0} = \frac{\frac{m}{M} + e^{-ct/\mu}}{\frac{m+M}{M}} = \frac{m + Me^{-ct/\mu}}{m+M}$$

$$v = v_0\left(\frac{m + Me^{-ct/\mu}}{m+M}\right)$$

As $t \to \infty$, $v \to \dfrac{v_0 m}{m+M}$

which is what we would have gotten by momentum conservation if $v = \mathcal{V}$

(4) Consider a mass density $\rho = \rho_0 e^{-ar}$ my first guess at the pre-ignition density of a star. Calculate the gravitational field.

Sln

Gauss' Law for Gravity

$$4\pi r^2 g = \oint \vec{g}\cdot\hat{n}\, dA = -4\pi G \cancel{\rho}\, M_{enc}$$

$$M_{enc} = \int_0^r 4\pi r^2 \rho(r)\, dr$$

$$= 4\pi\rho_0 \int_0^r r^2 e^{-ar^2} dr$$

$$= \frac{4\pi\rho_0}{a^3}\left(2 - \left(2 + 2ar + a^2r^2\right) e^{-ra}\right)$$

$$|\vec{g}| = -\frac{G M_{enc}}{r^2} = \frac{4\pi\rho_0}{a^3 r^2}\left[2 - \left(2 + ar + a^2r^2\right) e^{-ro}\right]$$

(5) A mass m travels in a decaying spiral orbit, $r(\theta) = r_0\left(1 - \frac{\theta}{2\pi}\right)$.

What force produces the orbit?

$$\frac{1}{r} = u = \frac{1}{r_0} \cdot \frac{1}{1 - c\theta} \qquad c = \frac{1}{2\pi}$$

$$\frac{du}{d\theta} = \frac{1}{r_0} \cdot c \cdot \frac{1}{(1-c\theta)^2}$$

$$\frac{d^2u}{d\theta^2} = \frac{2c^2}{r_0} \frac{1}{(1-c\theta)^3}$$

Differential eqn of orbit

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{m l^2 u^2} f(u^{-1})$$

$$f(u^{-1}) = -m l^2 u^2 \left[\frac{2c^2}{r_0} \frac{1}{(1-c\theta)^3} + \frac{1}{r_0} \frac{1}{(1-c\theta)} \right]$$

$$= -\frac{m l^2}{r^2} \left[\frac{2c^2 r_0^2}{r^3} + \frac{1}{r} \right]$$

Find $\theta(t)$? Central force so angular momentum is conserved.

$$L = m r^2 \dot{\theta} = m l$$

$$\frac{d\theta}{dt} = \dot{\theta} = \frac{l}{r^2} = l u^2 = \frac{l}{r_0^2} \frac{1}{1-c\theta}$$

$$\int_0^{\theta} 1-c\theta \, d\theta = \frac{l}{r_0^2} \int_0^t dt$$

$$\theta - \frac{c\theta^2}{2} = \frac{l}{r_0^2} t$$

$$\theta^2 - \frac{2}{c}\theta + \frac{2lt}{c r_0^2} = 0$$

$$\theta = \frac{2/c \pm \sqrt{\frac{4}{c^2} - \frac{8lt}{c r_0^2}}}{2}$$

$$= \frac{4\pi \pm \sqrt{16\pi^2 - 16\pi l t / r_0^2}}{2}$$

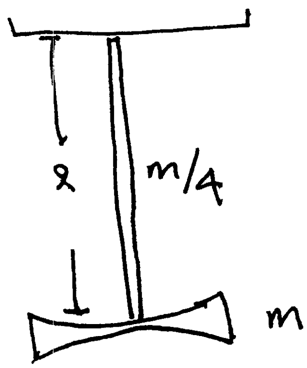
$$= 2\pi \left(1 \pm \sqrt{1 - lt/\pi r_0^2} \right)$$

When does particle reach origin?

$$\sqrt{} = 0 \Rightarrow \theta = 2\pi \Rightarrow r = 0$$

$$\frac{\ell t}{\pi r_0^2} = 1 \qquad t = \frac{\pi r_0^2}{\ell}$$

(6) An ornamental clock has a pendulum bob shaped as below



$\lambda = \gamma x^2$

The total mass of the support rod is $m/4$ and the total mass of the bob is m. The shape of the bob produces an effective mass density of $\lambda = \gamma x^2$ from $x = -a$ to a. Compute the period of the oscillation.

<u>Sln.</u> Moment of inertia about pivot

$$I = I_{rod} + I_{bob}$$

$$I_{rod} = \frac{Ml^2}{3} = \frac{ml^2}{12} \qquad \text{since } M = m/4$$

$$I_{bob} = I_{cm} + ml^2 \qquad \text{parallel-axis.}$$

$$I_{cm} = \int_{-a}^{a} x^2 dm = \gamma \int_{-a}^{a} x^4 dx \qquad dm = \gamma x^2 dx$$

$$= \gamma \frac{x^5}{5} \Big|_{-a}^{a} = \frac{2}{5} \gamma a^5$$

Find γ

$$m = \int_{-a}^{a} dm = \int_{-a}^{a} \gamma x^2 dx = \frac{2}{3} \gamma a^3$$

$$\gamma = \frac{3m}{2a^3}$$

$$I_{cm} = \frac{2}{5} \left(\frac{3m}{2a^3} \right) a^5 = \frac{3}{5} m a^2$$

$$I_{bob} = I_{cm} + m\ell^2 = \frac{3}{5} m a^2 + m\ell^2$$

$$I = I_{bob} + I_{rod} = \frac{3}{5} m a^2 + m\ell^2 + \frac{m\ell^2}{12}$$

$$= \frac{3}{5} m a^2 + \frac{13}{12} m\ell^2$$

EOM

$$N = \frac{dL}{dt} = I\ddot{\theta} = -mg\,l_{cm}\sin\theta$$

l_{cm} is distance from pivot to CM

$$l_{cm} = \frac{1}{\frac{5}{4}m}\left(\frac{m}{4}\cdot\frac{l}{2} + m l^2\right)$$

$$= \frac{4}{5}\left(\frac{9}{8}l\right) = \frac{9}{10}l$$

Linearize

$$I\ddot{\theta} + mg\,l_{cm}\,\theta = 0$$

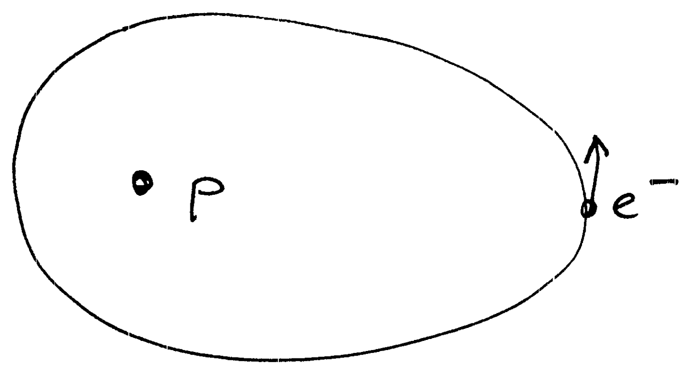
$$\ddot{\theta} + \frac{mg\,l_{cm}}{I}\theta = 0$$

$$\omega^2 = \frac{mg\,l_{cm}}{I} = \frac{mg\left(\frac{9}{10}l\right)}{\frac{3}{5}ma^2 + \frac{13}{12}ml^2}$$

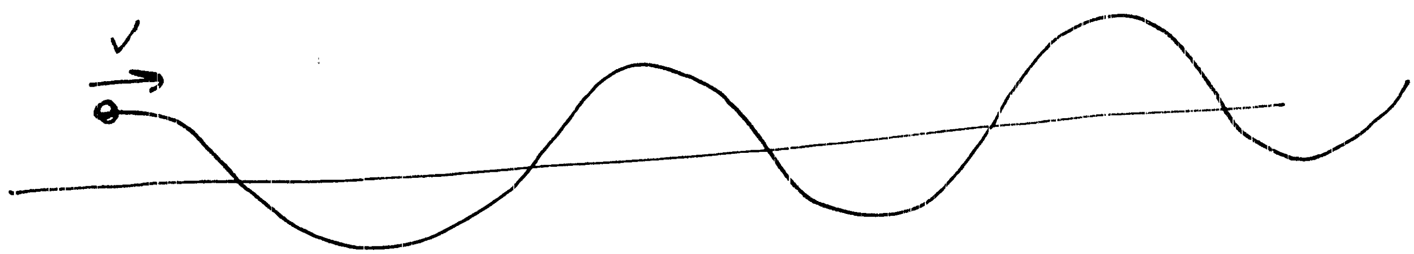
If $a = l/10$

$$\omega^2 = \frac{g}{l}(9/10)\Big/\left(\frac{3}{100} + \frac{13}{12}\right) = 0.81\,g/l$$

(7) I was considering something with the orbits of the classical hydrogen atom



(8) I was also considering a pure kinematics problem



compute normal force as a function of x, but it looked annoying.