Generalized Forces

Work with Lagrange's eqn for a moment

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

$$\frac{\partial T}{\partial q_i} - \frac{\partial V}{\partial q_i} = \dot{p}_i$$

generalized momentum

If $q_i = x, y, z$, then the conservative force resulting from the potential is

$$F_x = \frac{-\partial V}{\partial q_x}$$

It would then be reasonable to identify a generalized force

$$Q_i = -\frac{\partial V}{\partial q_i}$$

Note, the term $\frac{\partial T}{\partial q_i}$ is often but not necessarily zero. This term contains <u>fictitious</u> forces associated with the movement of the coordinate system.

Now, if we wanted to introduce a non-conservative force, it should act like the conservative force once we have an equation of motion. Now, let Q_i be generalized forces not derivable from a potential, for example either non-conservative or constraint forces.

$$\frac{\partial L}{\partial q_i} + Q_i = \dot{p}_i$$

$$Q_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}$$

Relating Coordinate Forces to Generalized Forces

Suppose we have a force $\vec{F} = (F_x, F_y, F_z)$, how do we express it as a generalized force?

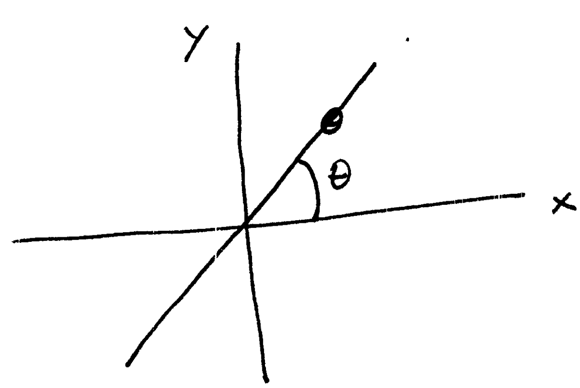
The generalized coordinates are actually defined by coordinate transforms

$$x_1 = x_1(q_1, q_2 \cdots q_n)$$

$$x_2 = x_2(q_1, q_2, \cdots q_n)$$

For example, the coordinates of r of a bead constrained to a ~~wi~~ wire is defined by

$$x = r\cos\theta \qquad y = r\sin\theta \qquad \theta \text{ fixed}$$



The transformation to a generalized force is very much like a coordinate transform.

$$Q_i = \sum_j F_j \frac{\partial x_j}{\partial q_i}$$

So for the bead on a wire, a Hooke's law force $\vec{F} = -kx\hat{x} - ky\hat{y}$ becomes

$$Q_r = F_x \frac{\partial x}{\partial r} + F_y \frac{\partial y}{\partial r}$$

$$= -kx\cos\theta - ky\sin\theta$$

$$= -kr\left(\cos^2\theta + \sin^2\theta\right) = -kr$$

But its not quite that simple. Suppose we have a bead constrained to a circle which experiences a constant force along the circle (like the test problem).

$$\vec{F} = F_0\,\hat{\theta} = -F_0\sin\theta\,\hat{x} + F_0\cos\theta\,\hat{y}$$

from Griffith's back cover

$$Q_\theta = F_x \frac{\partial x}{\partial \theta} + F_y \frac{\partial y}{\partial \theta}$$

$$= - F_x r \sin\theta + F_y r \cos\theta$$

$$= + F_0 r \sin^2\theta + F_0 r \cos^2\theta$$

$$= F_0 r = \text{torque.}$$

So the generalized forces associated with angular variables are torques. This makes sense because the momenta associated are angular momenta

so $$\frac{\partial L}{\partial \theta} + Q_\theta = \dot{P}_\theta$$

says the torque equals the time rate of change of angular momentum.

<u>Ex</u> Bead constrained to rod, Hooke's Law force, linear frictional drag.

$$Q_r = -c\dot{r} \qquad L = \tfrac{1}{2}m\dot{r}^2 - \tfrac{1}{2}kr^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = Q_r$$

<u>Ex</u> Bead constrained to circle, constant drag force

$$Q_\theta = -F_\theta a \,\mathrm{sign}(\dot{\theta})$$

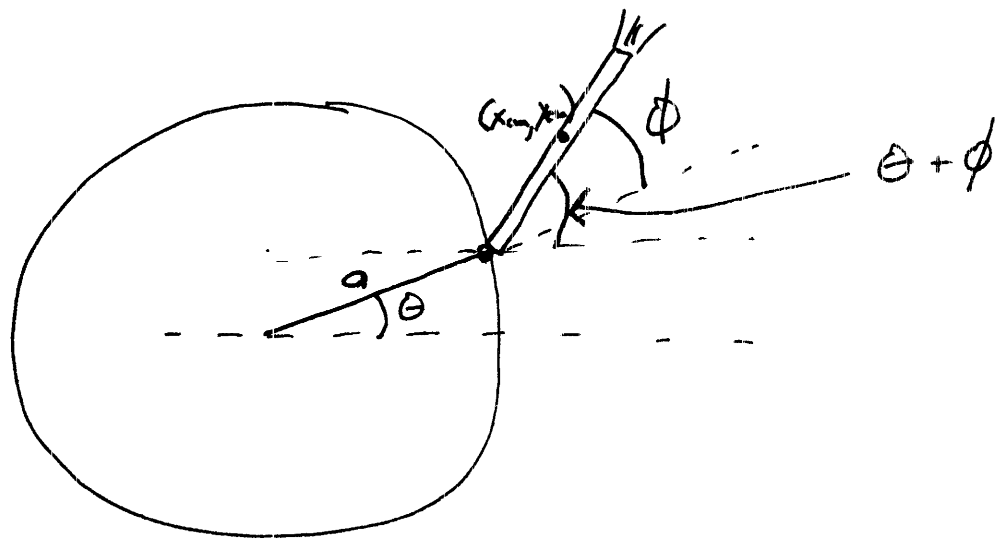
$$L = \tfrac{1}{2}m(a\dot{\theta})^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q_\theta$$

$$\underbrace{ma^2\ddot{\theta}} = -F_\theta a \qquad \text{if } \dot{\theta} > 0$$

$$I\dot{\omega} = \tau$$

Ex Rocket with thrust F_0 attached to disk with moment of inertia I_{disk} pivoted to rotate about center. Model rocket as rod with mass m and length $2l$.



Generalized Force is a torque in $-\hat{\theta}$ direction

$$Q_\theta = -F_0 a \cos\phi$$

$$x_{cm} = a\cos\theta + l\cos(\theta+\phi)$$

$$y_{cm} = a\sin\theta + l\sin(\theta+\phi)$$

$V = 0$

$$T = \frac{1}{2} I_{disk} \dot{\theta}^2 + \frac{1}{2} m \left(\dot{x}_{cm}^2 + \dot{y}_{cm}^2 \right) + \frac{1}{2} I_{rod} (\dot{\phi} + \dot{\theta})^2$$

ϕ eom

$$\frac{\partial L}{\partial \phi} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = 0$$

θ eom

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = -Q_\theta = F_0 \, a \cos \phi$$

In essence, we have reduced the number of variables by solving f for q_i and the using it to eliminate some degrees of freedom.

We could accomplished the application of the constraint in an alternate method by using Lagrange Multipliers.

<u>Ex</u> Consider the problem of findings the point on a line nearest the origin. We want to minimize $F(x,y) = x^2+y^2$ under the constraint $y = 2x+1$ or $2x - y + 1 = 0 \equiv f$. We could substitute the equation for the line and have a one-dimensional problem. However, we could also consider minimizing

$$F' = F + \lambda f = x^2+y^2 + \lambda(2x-y+1) = 0$$

If the constraint is satisfied $F' = F$. λ is called a Lagrange multiplier.

Lagrange Multipliers

Our remaining tasks before returning to Newton is to deal with the forces of constraint hidden in Lagranges equations and to find some way to introduce non-conservative forces into the equations.

We have been working with system whose motion is constrained, not free to fully explore 3-space. These constraints, $f(\{x_i\},t)=0$, mean the 3N coordinates needed to describe N particles are not independant. Instead of using the 3N spacial coordinates, we have selected a restricted set of generalized coordinates $\{q_i\}$ that are independant and automatically satisfy the constraints. The normal 3-space coordinates are recovered by

$$x_i = x_i(q_1, q_2 \cdots q_n, t)$$

$$\frac{\partial F'}{\partial x} = 2x + 2\lambda = 0 \qquad \frac{\partial F'}{\partial y} = 2y - \lambda = 0$$

$$y = 2x + 1$$

Solve 3 eqns in 3 unknowns

$$4x + 2 = \lambda$$

$$2x + 2(4x+2) = 0$$

$$x = -2/5$$

$$y = 1/5$$

$$\lambda = 2/5$$

We will apply the method of Lagrange multipliers to impose the physical constraints f and to extract forces of constraint.

Using Lagrange multipliers to extract forces of constraint.

A constraint equation relates coordinates effectively lowering the degrees of freedom of a system. Consider, as the book does, two generalized coordinates connected by the constraint $f(q_1, q_2, t) = 0$.

The constraint couples variations of the two coordinates

$$\delta f = \frac{\partial f}{\partial q_1}\delta q_1 + \frac{\partial f}{\partial q_2}\delta q_2 = 0$$

$$\delta q_2 = \frac{\dfrac{\partial f}{\partial q_1}}{\dfrac{\partial f}{\partial q_2}}$$

As we take the variation of the ~~Lagr~~ action

$$\delta \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right]\delta q_i\,dt$$

$$\delta \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} \left[\left(\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} \right) - \left(\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} \right) \frac{\frac{\partial f}{\partial q_1}}{\frac{\partial f}{\partial q_2}} \right] \delta q_1\, dt$$

$$= 0$$

<u>Cross-multiplying</u>

$$\frac{\frac{\partial L}{\partial q_1} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1}}{\frac{\partial f}{\partial q_1}} = \frac{\frac{\partial L}{\partial q_2} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2}}{\frac{\partial f}{\partial q_2}} = -\lambda(t)$$

at all times. The only way this can happen is if both independent equal the same function of time $-\lambda(t)$. This is basically the separation of variables argument.

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \lambda(t)\frac{\partial f}{\partial q_i} = 0$$

The last term looks like a generalized force and is actually the force of constraint

$$Q_i = \lambda(t)\frac{\partial f}{\partial q_i}$$

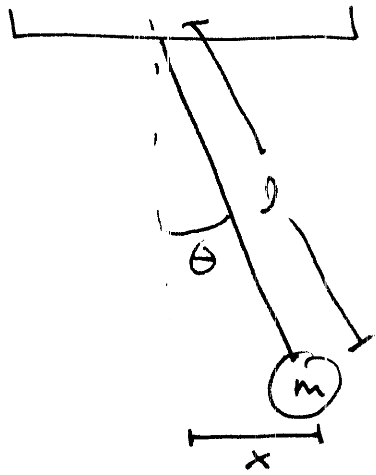
in the direction associated with the coordinate q_i

Recall, the coordinate forces are related through the generalized forces through

$$\cancel{Q_i = \sum_j F_{x_i} \frac{\partial x_i}{\partial}}$$

$$Q_i = \sum_j F_{x_j} \frac{\partial x_j}{\partial q_i}$$

Ex Pendulum



Equation of constraint

$$l = \sqrt{x^2+y^2}$$

$$f = \sqrt{x^2+y^2} - l = 0$$

Now, work Lagrangian with both x and y and impose constraint with Lagrange multiplier.

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \qquad V = mgy$$

$$L = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2\right) - mgy$$

x EOM

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \lambda(t)\frac{\partial f}{\partial x} = 0$$

y EOM

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} + \lambda(t)\frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{l} \qquad \frac{\partial f}{\partial y} = \frac{y}{l}$$

<u>x eom</u>

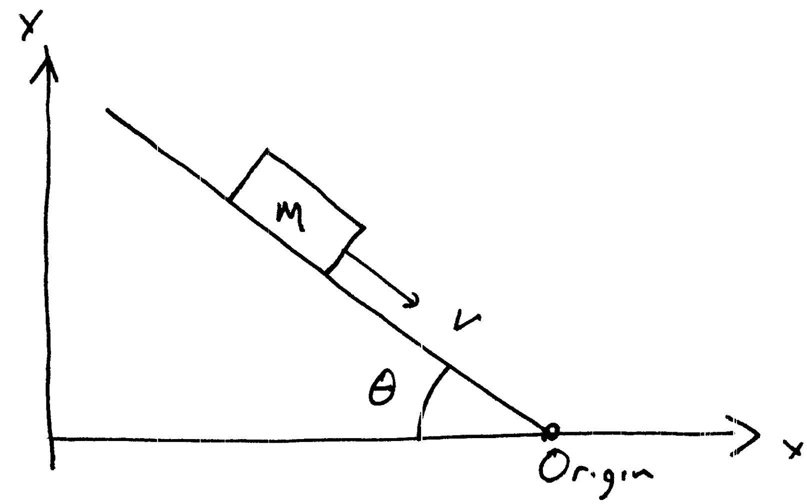
$$-m\ddot{x} + \lambda(t)\frac{x}{l} = 0$$

<u>y eom</u>

$$-mg - m\ddot{y} + \lambda(t)\frac{y}{l} = 0$$

We would then solve these with the constraint equation to fing $\lambda(t)$ = tension.

$\underline{\text{Ex}}$ Mass sliding down frictionless plane



Work problem with both x and y and use Lagrange multiplier to impose constraint, $y = -\tan\theta\, x$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 \qquad V = mgy$$

$$L = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) - mgy$$

$\underline{x \text{ EOM}}$

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} + \lambda \frac{\partial f}{\partial x} = 0$$

$$f = y + \tan\theta\, x = 0$$

$$\frac{\partial f}{\partial x} = \tan\theta \qquad \frac{\partial f}{\partial y} = 1$$

$$-m\ddot{x} + \lambda \tan\theta = 0$$

<u>y EOM</u>

$$-mg - m\ddot{y} + \lambda = 0$$

<u>constraint</u> $\ddot{y} = -\tan\theta\,\ddot{x}$

<u>Substitute into y EOM</u>

$$-mg - m\left(-\tan\theta\,\ddot{x}\right) + \lambda = 0$$

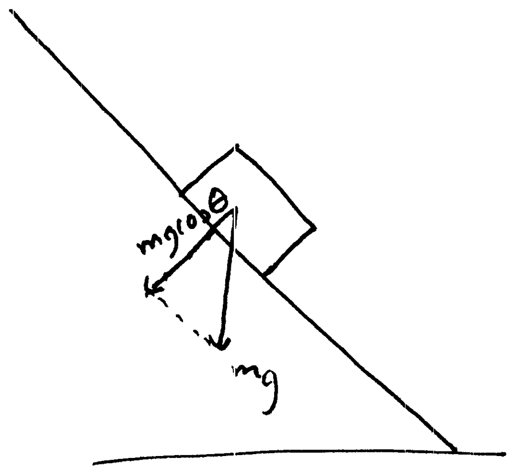
<u>Substitute x EOM</u>

$$-mg + m\tan\theta\left(\frac{\lambda\tan\theta}{m}\right) + \lambda = 0$$

$$\lambda\left(1+\tan^2\theta\right) = mg \qquad 1+\tan^2\theta = \sec^2\theta$$

$$\lambda = mg\cos^2\theta$$

Force of Constraint



$|F_N| = mg\cos\theta$ $\qquad$ $\vec{F}_N = mg\cos\theta(\sin\theta, \cos\theta)$

Force of Constraint from Lagrange Multipliers

$$\vec{Q} = \left(\lambda \frac{\partial f}{\partial x}, \lambda \frac{\partial f}{\partial y}\right)$$

$$= mg\cos^2\theta(\tan\theta, 1)$$

$$= mg\cos\theta(\sin\theta, \cos\theta)$$