Gravitation

<u>Newton's Law of Gravitation</u> - A mass m_1 exerts a force $\vec{F}_{12}$ on a mass m_2

$$\vec{F}_{12} = -\frac{G m_1 m_2}{r_{12}^2} \hat{r}_{12}$$

where $\vec{r}_{12}$ is the displacement from m_1 to m_2

$$G = 6.67259 \times 10^{-11} \frac{Nm^2}{kg^2}$$

If $\vec{r}_1$ and $\vec{r}_2$ are the positions of the two masses $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$

and $\hat{r}_{12} = \frac{\vec{r}_{12}}{r_{12}}$

Gravitational Field ($\vec{g}$) - Force per unit mass a mass would feel if placed in the field

$$\vec{g}_P = \frac{\vec{F}}{m} = \frac{-Gm}{r_{1P}^2}\hat{r}_{1P}$$

is the gravitational field at point P due to a mass m, at $\vec{r}_1$.

Gravitational Potential (Φ)

$$\vec{g} = -\nabla\Phi$$

- For a mass m ~~at~~

$$\Phi = -\frac{Gm}{r}$$

Gravitational Potential Energy (V)

$$V = M\Phi$$

the potential energy ~~due~~ of mass M due to the gravitational field of mass m.

The work to move a mass M from point A to point B (if KE constant) is the change in potential energy, ΔU_{AB}

$$W_{AB} = \Delta V_{AB} = \cancel{M(V_B}$$

$$= V_B - V_A$$

$$= M\left(\Phi_B - \Phi_A\right)$$

<u>Ex</u> How much work is required to move a $M = 1\,kg$ mass from $a\hat{x}$ to $b\hat{y}$ where $a = 1m$ and $b = 2m$? in the field of a $2kg = m$ mass?

we do work

$$W_{AB} = M\left(\Phi_B - \Phi_A\right) = -\int_{A \to B} \vec{F} \cdot d\vec{l}$$

$$= M\left(\frac{-mG}{b} - \frac{-mG}{a}\right)$$

$$= mMG\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$= (1\,kg)(2\,kg)\left(6.67 \times 10^{-11} \frac{Nm^2}{kg^2}\right)\left(\frac{1}{1m} - \frac{1}{2m}\right)$$

$$= 6.67 \times 10^{-11}\,J$$

Ex Compute the energy of two 1kg masses placed a distance $d = 1m$ apart.

E The energy of the system is the work to build it. Let W_i be the work to place mass i.

$$U = W = W_1 + W_2$$

The work to place the first mass is zero $W_1 = 0$

The work to place the second mass

$$W_2 = m_2\left(-\frac{m_1 G}{d}\right)$$

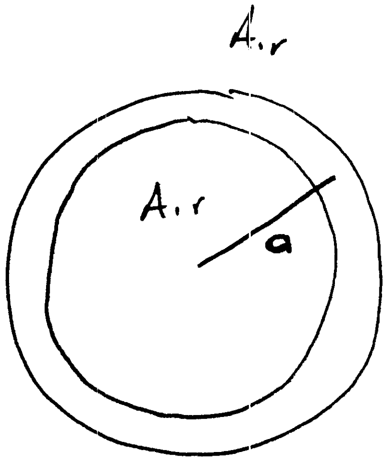
$$= -\frac{m_1 m_2 G}{d}$$

$$= -\frac{(1\,kg)(1\,kg)\left(6.67\times10^{-11}\,\frac{Nm^2}{kg^2}\right)}{1m}$$

$$= -6.67\times10^{-11}\,J$$

$$V = W_1 + W_2 = -6.67\times10^{-11}\,J$$

Ex Compute field of thin spherical shell of total mass M, and radius a.



Method I - Integrate $\vec{g} = \frac{-MG}{r^2}\hat{r}$ over mass.

Method II Integrate $\Phi = \frac{-MG}{r}$ over mass and differentiate.

Method III Gauss for gravity.

$$\oint_S \vec{g} \cdot \hat{n}\, dA = \frac{-M_{enc}}{\varepsilon_g} \qquad \cancel{M_{en}}$$

If $r > a$, $M_{enc} = M$. Select spherical surface of radius r, by symmetry $\vec{g} \parallel \hat{n}$, $\Rightarrow \vec{g} \cdot \hat{n} = g$.

g is constant on spherical surface

$$\oint \vec{g} \cdot \hat{n}\, dA = g \int dA = 4\pi r^2 g$$

	Gravity	Electrostatics
Force	$\vec{F}_{12} = \dfrac{-Gm_1m_2}{r_{12}^2}\,\hat{r}_{12}$	$\vec{F}_{12} = \dfrac{kq_1q_2}{r_{12}^2}\,\hat{r}_{12}$
Potential	$\Phi = \dfrac{-Gm}{r}$	$\Phi = \dfrac{kq}{r}$
Field	$\vec{g}_{1P} = \dfrac{\vec{F}_{1P}}{m}$	$\vec{E}_{1P} = \dfrac{\vec{F}_{1P}}{Q}$
	$= \dfrac{-Gm_1}{r_{1P}^2}\,\hat{r}_{1P}$	$= \dfrac{kq_1}{r_{1P}^2}\,\hat{r}_{1P}$
	$G = \dfrac{1}{4\pi\varepsilon_g}$	$k = \dfrac{1}{4\pi\varepsilon_0}$
Gauss	$\oint \vec{g}\cdot\hat{n}\,dA = -\dfrac{M_{enc}}{\varepsilon_g}$	$\oint \vec{E}\cdot\hat{n}\,dA = \dfrac{Q_{enc}}{\varepsilon_0}$
	$= -4\pi G M_{enc}$	
	$\nabla\cdot\vec{g} = -\dfrac{\rho_m}{\varepsilon_g}$	$\nabla\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}$

Gauss' Law for Gravity (Spherical)

$$4\pi r^2 g = \oint_S \vec{g} \cdot \hat{n}\, dA = \frac{-M_{enc}}{\varepsilon_g}$$

$$g = \frac{-M_{enc}}{4\pi\varepsilon_g r^2} \qquad \text{radially outward}$$

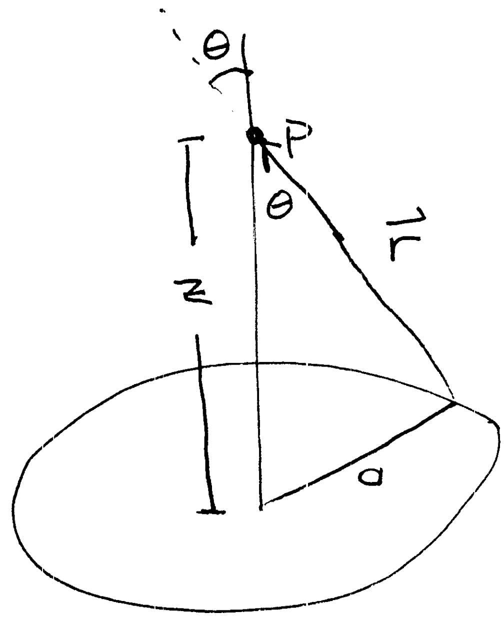
For $r < a$, $M_{enc} = 0$, $g = 0$

For $r > a$ $\quad M_{enc} = M$

$$g = \frac{-M}{4\pi\varepsilon_g r^2} = -\frac{GM}{r^2}$$

$$\vec{g} = -\frac{GM}{r^2}\hat{r}$$

Ex Thin ring of mass with uniform linear mass density λ and radius a. Compute Φ and $\vec{g}$ along axis.



The component of the ~~magn~~ gravitational field along the z-axis is

$$d\vec{g} = - \frac{G\,dM}{r^2} \cos\theta \,\hat{z}$$

where dM is a small piece of mass.

$$dM = \lambda a \, d\theta$$

By symmetry, the total field is along the z-axis so this is the only component that matters.

$$r = \sqrt{z^2 + a^2}$$

$$\cos\theta = \frac{z}{r} = \frac{z}{\sqrt{a^2+z^2}}$$

$$d\vec{g} = \frac{-G(\lambda a\, d\theta)}{\left(\sqrt{z^2+a^2}\right)^2} \cdot \frac{z}{\sqrt{a^2+z^2}}\,\hat{z}$$

$$= \frac{-G\lambda a z\, d\theta}{(z^2+a^2)^{3/2}}\,\hat{z}$$

The total field is the integral of $d\vec{g}$ around the ring.

$$\vec{g} = \int_0^{2\pi} \frac{-G\lambda a z\, d\theta}{(z^2+a^2)^{3/2}}\,\hat{z}$$

$$= -\frac{G\lambda a z\hat{z}}{(z^2+a^2)^{3/2}} \int_0^{2\pi} d\theta$$

$$= \frac{-G(2\pi a\lambda)\, z}{(z^2+a^2)^{3/2}}\,\hat{z}$$

The total mass is $M = 2\pi a \lambda$

$$\vec{g} = -\frac{GM}{r^2}\cos\theta\,\hat{z} = -\frac{GMz}{(z^2+a^2)^{3/2}}\,\hat{z}$$

Since $\vec{g}$ only depends on r along the axis

$$\Phi(z) = -\int g\,dz$$

$$= -\frac{MG}{\sqrt{z^2+a^2}} + C$$

$$= -\frac{MG}{r} + C$$

If $\Phi(a)=0$, $C=0$.

<u>Ex</u> Compute Φ directly.

$$\Phi = \int d\Phi = -\int \frac{G\,dM}{r}$$

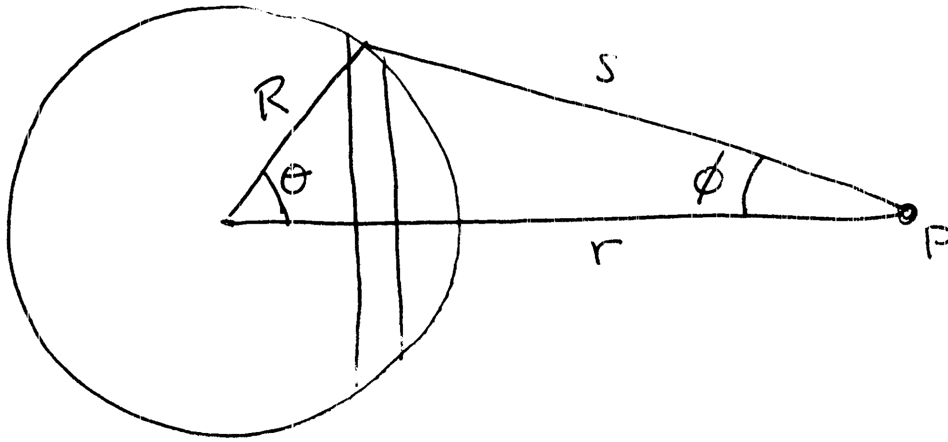
$$= -\int_0^{2\pi} \frac{G(\lambda a\,d\theta)}{\sqrt{z^2+a^2}}$$

$$= \frac{-G\lambda a}{\sqrt{z^2+a^2}} \int_0^{2\pi} d\theta$$

$$= \frac{-(2\pi\lambda a)G}{\sqrt{z^2+a^2}} = \frac{-GM}{\sqrt{z^2+a^2}}$$

Ex

Thin disk with surface mass density σ



$$\Phi = -\int \frac{G\,dM}{s}$$

$$dM = \sigma 2\pi (R\sin\theta) R\,d\theta$$

$$= 2\pi\sigma R^2 \sin\theta\, d\theta$$

By the law of cosines

$$s^2 = R^2 + r^2 - 2rR\cos\theta$$

$$2s\,ds = 2rR\sin\theta\, d\theta$$

$$d\theta = \frac{s\,ds}{rR\sin\theta}$$

$$\Phi = -\int \frac{G\,dM}{s}$$

$$= -G \int \frac{2\pi\sigma R^2 \sin\theta \, d\theta}{s}$$

$$= -2\pi\sigma G R^2 \int \frac{\sin\theta}{s} \left(\frac{s\,ds}{rR\sin\theta} \right)$$

$$= -\frac{2\pi\sigma G R^2}{rR} \int_{r-R}^{r+R} ds$$

$$= -\frac{2\pi\sigma G R}{r} \underbrace{\left(r+R - (r-R) \right)}_{2R}$$

$$= -\frac{4\pi\sigma G R^2}{r}$$

$$= -\frac{MG}{r} \qquad M = -4\pi R^2 \sigma$$