Rockets

* Up until now when we talked about a change in momentum, we restricted our attention to changes in velocity
* In the more general case a change in momentum involve both a change in mass and a change in velocity

Examples:

1. Asteroid moving and burning thru atmosphere
2. Snow ball
3. Rockets
4. Octopus

Rockets

Assume a rocket with an initial mass M_0 and a constant rate of change of mass α (in units of mass/time) in which the emitted gas has a velocity V relative to the rocket.

The mass of the rocket + gas system during the acceleration phase:

$$M(t) = M_0 - \alpha t \quad \text{(while there is still gas left)}$$

2. To find an equation for the velocity of the rocket as a function of time, we will use conservation of momentum;

$$\frac{dP_{TOTAL}}{dt} = F_{ext} = 0 \quad \text{(There are no external forces acting on the sys.}$$

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$$\frac{dP_{ROCKET}}{dt} + \frac{dP_{GASS}}{dt} = 0 \quad (P_{TOTAL} = P_{ROCKET} + P_{GASS})$$

$$\frac{d}{dt}\left(M(t)\, v\right) = M\dot{v} + \dot{M}v = \underbrace{M\dot{v}}_{\text{change in } P \text{ due to change in velocity}} - \underbrace{\alpha v}_{\text{change in } P \text{ due to change in mass}}$$

3. The change in momentum of the gas; The change in P_{gass} has 2 components: (1) change in P_{gass} due to a change in momentum of the gass due to a change in its velocity

(2) a change in momentum due to a change (reduction) in the amount of gas.

1. The first component (1) is zero since the gas does not accelerate after it has been emmited.

2. The second term (2) is given by:

$\Delta m = \alpha \Delta t$: The amount of gas emitted over a time Δt.

$v_{gass} = v - V$: Velocity of gass in the laboratory coordinates given that the velocity of the gass relative to the rocket is V.

$\Delta P_{gass} = \Delta m (v - V) = \Delta t \, \alpha (v - V)$: change in momentum

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$$\frac{d P_{gass}}{dt} = \alpha (v - V)$$

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$$M \dot{v} - \alpha v + \alpha (v - V) = 0$$

$$\Downarrow$$

$M \dot{v} = \alpha V$ substituting this equation into the expression for $M(t)$:

$$(M - \alpha t) \frac{dv}{dt} = \alpha V$$

$$\Downarrow$$

$\alpha V \dfrac{dt}{M_0 - \alpha t} = dv$: seperation of variables

$$\alpha V \int_{t=0}^{t=t} \frac{dt}{M_0 - \alpha t} = \int_{v(0)}^{v(t)} dv$$

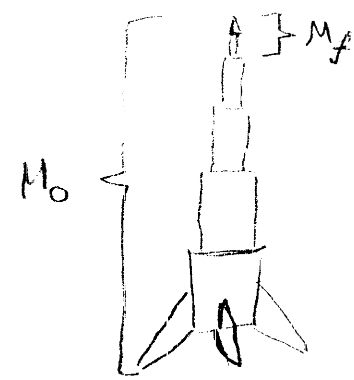
$$- \alpha V \frac{1}{\alpha} \ln (M_0 - \alpha t) \Big|_{t=0}^{t=t} = v(t) - v(0)$$

$$\boxed{v(t) = v(0) + V \ln \frac{M_0}{M_0 - \alpha t}}$$

(a) The velocity of the rocket when all the gass has been spent does not depend on α (emmision rate), but only on V and the ratio : M_0/M_f (where M_f is the mass of the rocket w/o the gass).

(b) It is relatively easy to reach a speed of V but in order to reach speeds larger than V we must have a very large ratio M_0/M_f (b/c the ln term grows very slowly). That explains why space launchers have several phases. This way we start with a very large M_0 compared to M_f.

(c) V is characteristic of the chemical reaction (burning):



$\frac{1}{2} m V^2 = m \mathcal{E}$ amount of energy released per unit mass.

$V = \sqrt{2\mathcal{E}}$

For $2H_2 + O_2 \to 2H_2O$; $\mathcal{E} = 1.3 \cdot 10^{11}$ erg/gr

$V \doteq 5$ km/sec assuming maximal yield. In reality though $V \sim 3$ km/sec

(d) To reach $v_{escape} = 11$ km/sec we need $M_0/M_f \approx 50$

7. Now let's take gravity into account. Gravity exerts an external force on the system. Thus the emmited gas is not moving with a constant velocity anymore but it accelerates!!!

$$\frac{dP_{gass}}{dt} = \underbrace{\alpha(v-V)}_{\substack{\text{change in}\\ \text{momentum}\\ \text{from change}\\ \text{in amount}\\ \text{of gass}}} - \underbrace{\overbrace{\alpha t}^{\text{amount of gass}} \cdot g}_{\substack{\text{change in}\\ \text{momentum}\\ \text{(force) from}\\ \text{acceleration}\\ \text{of the gas}}}$$

$$\frac{dP_{rocket}}{dt} + \frac{dP_{gass}}{dt} = F_{ext}$$

$$M\dot{v} - \alpha v + \alpha(v-V) - \alpha t g = -M_0 g$$

$$\underbrace{(M_0 - \alpha t)}_{\text{Instantaneous mass}} \underbrace{\frac{dv}{dt}}_{a} = \underbrace{-(M_0 - \alpha t)g}_{\text{instantaneous } Mg} + \underbrace{\alpha V}_{\text{Thrust}}$$

8. We can already see (before even starting to solve analytically) that now the rocket will not start raising if: $M_0 g > \alpha V$

9. Thus, in order for the rocket to start moving upward the thrust must be larger then the weight of the rocket.

10.

$$\int_{v(t=0)}^{v(t)} dv = \int_{t=0}^{t} \left(\frac{-(M_0 - \alpha t)g + \alpha V}{M_0 - \alpha t} \right) dt$$

$$= \int_{t=0}^{t} -g\,dt + \int_{t=0}^{t} \frac{\alpha V}{M_0 - \alpha t}\,dt$$

$$\Downarrow$$

$$\boxed{v(t) = v(0) - gt + V \ln \frac{M_0}{M_0 - \alpha t}}$$