Homework 10 - Last Homework

Due Tuesday 12/8/2009 at 5:30pm in my box in physics. These may also be handed in at the end of Justin Mitchell's office hours in PHYS 228 from 4:00-5:30pm Tuesday.

Fowles Problems

8.2

8.6

8.8

8.9

8.12

8.16

8.20

8.22

8.2

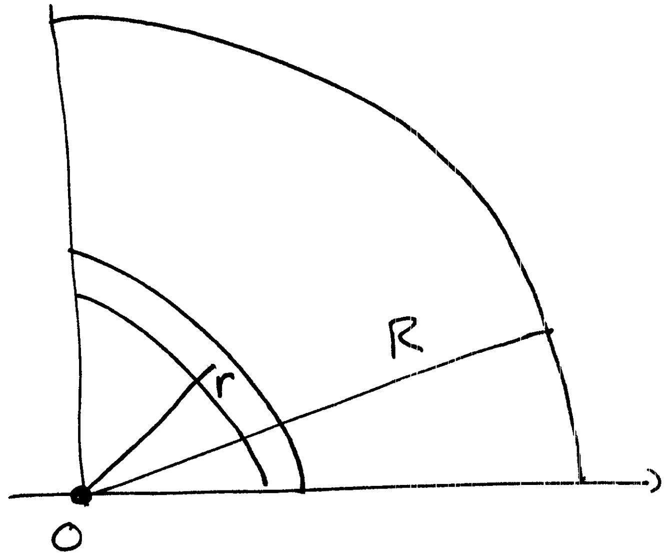
$\lambda = cx \qquad dm = \lambda\,dx = cx\,dx$

$$r_{cm} = \frac{\int_0^b x\,dm}{\int_0^b dm}$$

$$= \frac{\int_0^b cx^2\,dx}{\int_0^b cx\,dx} = \frac{\frac{b^3}{3}}{\frac{b^2}{2}}$$

$$= \frac{2}{3}b$$

8.6



Divide the sphere into cylindrical shells about the O axis.

The volume of a shell is

$$dV = L\left(\frac{2\pi r}{4}\right) dr$$

where L is the distance from the plane of the page to where the shell interects the sphere.

$$L^2 + r^2 = R^2$$

$$L = \sqrt{R^2 - r^2}$$

$$dm = m\,dV = \left(\frac{\pi r}{2}\right)\sqrt{R^2 - r^2}\,\rho\,dr$$

where ρ is the mass density.

$$\rho = \frac{m}{V} = \frac{8m}{\frac{4}{3}\pi R^3} = 6\frac{m}{\pi R^3}$$

The moment of inertia about the O axis is

$$I_0 = \int_{\text{solid}} r^2 dm$$

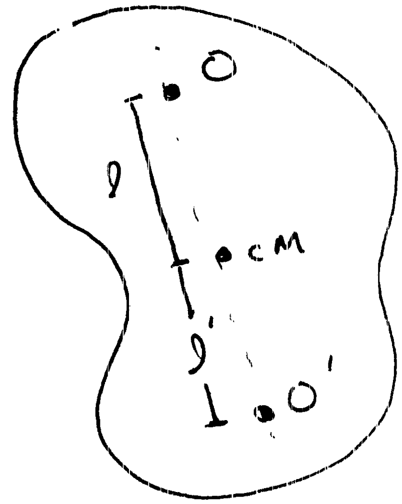
$$= \frac{\pi\rho}{2} \underbrace{\int_0^R dr\, r^3 \sqrt{R^2 - r^2}}_{\frac{2}{15}R^5 \text{ (maple)}}$$

$$I_0 = \left(\frac{\pi\rho}{2}\right)\left(\frac{2}{15}R^5\right) = \frac{\pi\rho}{15}R^5$$

<u>Substituting ρ</u>

$$I_0 = \left(\frac{\pi R^5}{15}\right)\left(\frac{6m}{\pi R^3}\right) = \frac{2}{5}mR^2$$

8.8



$$T = 2\pi \sqrt{\frac{I_0}{mgl}}$$

$I_0 \equiv$ Moment of Inertia about O

$$d \equiv l + l'$$

Center of Oscillation (O') –

$$l l' = k_{cm}^2$$

$$I_{cm} = m k_{cm}^2$$

Parallel Axis Thm

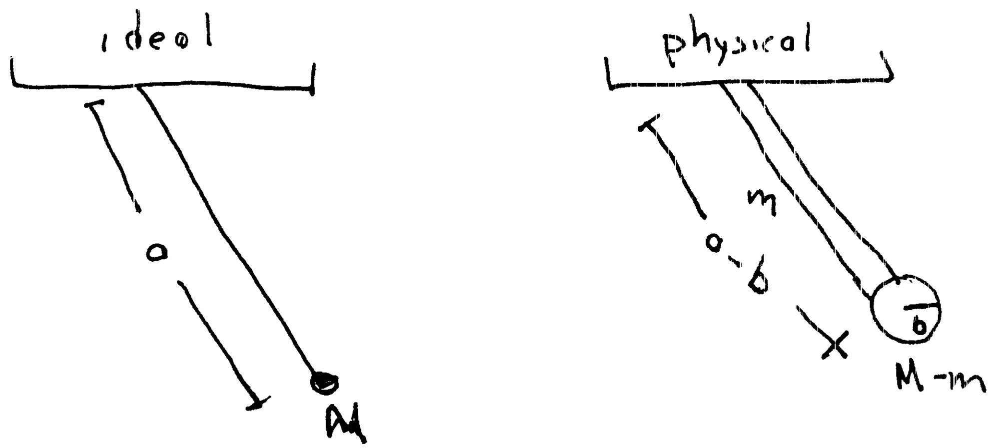
$$I_0 = I_{cm} + m l^2 = m\left(k_{cm}^2 + l^2\right)$$

$$= m\left(l l' + l^2\right) = m l \left(l + l'\right)$$

$$= m l d$$

$$T = 2\pi\sqrt{\frac{I_0}{mgl}} = 2\pi\sqrt{\frac{mld}{mgl}} = 2\pi\sqrt{\frac{d}{g}}$$

8.9



Ideal

$$T_i = 2\pi\sqrt{\frac{a}{g}}$$

Physical

$$T_p = 2\pi\sqrt{\frac{I}{Mg\,l_{cm}}}$$

Calculate I

$$I = I_{rod} + I_{ball}$$

Rod about end (wiki)

$$I_{rod} = \frac{ml^2}{3} = \frac{m(a-b)^2}{3}$$

$$I_{ball} = I_{cm} + ml^2 \quad \text{(parallel axis)}$$

$$= \frac{2}{5}mr^2 + ml^2$$

$$I_{ball} = \frac{2}{5}(M-m)b^2 + (M-m)a^2$$

$$I = \frac{m(a-b)^2}{3} + (M-m)a^2 + \frac{2}{5}(M-m)b^2$$

$$= \frac{m}{3}(a-b)^2 + (M-m)\left(a^2 + \frac{2}{5}b^2\right)$$

<u>Find CM physical pendulum</u>

$$l_{cm} = \frac{1}{M}\sum_i m_i \vec{r_i} = \frac{1}{M}\left(m r_{cm}^{rod} + (M-m) r_{cm}^{ball} \right)$$

$$= \frac{1}{M}\left(m\left(\frac{a-b}{2}\right) + (M-m)a \right)$$

$$= \frac{1}{M}\left(Ma - \frac{m}{2}(a+b) \right)$$

$$= a - \frac{1}{2}\frac{m}{M}(a+b)$$

Ratio of periods

$$\frac{T_p}{T_i} = \frac{2\pi\sqrt{\frac{I}{M g l_{cm}}}}{2\pi\sqrt{a/g}} = \sqrt{\frac{I}{a\, l_{cm} M}}$$

$$= \left(\frac{\frac{m(a-b)^2}{3} + (M-m)a^2 + \frac{2}{5}(M-m)b^2}{Ma^2 - \frac{ma}{2}(a+b)} \right)^{1/2}$$

(b) Using $M = 1\,kg$, $m = 0.01\,kg$, $a = 1.27\,m$, $b = 0.05\,m$

and Maple

$$T_p / T_i = 0.99944$$

> $rat := \left(\dfrac{\left(\dfrac{m \cdot (a-b)^2}{3} + (M-m) \cdot a^2 + \left(\dfrac{2}{5}\right) \cdot (M-m) \cdot b^2 \right)}{M \cdot a^2 - \left(\dfrac{m \cdot a}{2}\right) \cdot (a+b)} \right)^{\left(\frac{1}{2}\right)}$

$$rat := \sqrt{\frac{\frac{1}{3}\, m\, (a-b)^2 + (M-m)\, a^2 + \frac{2}{5}\, (M-m)\, b^2}{M a^2 - \frac{1}{2}\, m\, a\, (a+b)}} \tag{1}$$

> $simplify(rat)$

$$\frac{1}{15} \sqrt{30} \sqrt{\frac{10\, a^2 m + 10\, m\, a\, b + b^2 m - 15\, M a^2 - 6\, b^2 M}{a\, (-2\, M a + m a + m b)}} \tag{2}$$

> $M := 1$

$$M := 1 \tag{3}$$

> $m := 0.01$

$$m := 0.01 \tag{4}$$

> $a := 1.27$

$$a := 1.27 \tag{5}$$

> $b := 0.05$

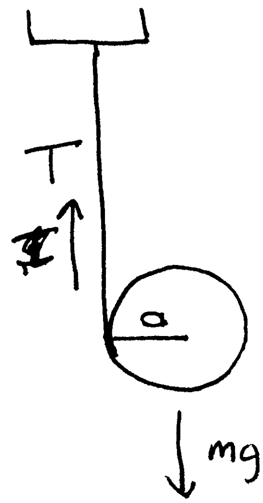
$$b := 0.05 \tag{6}$$

> rat

$$0.9994402775 \tag{7}$$

>

8.12



Let counterclockwise rotation be positive

No slipping so energy is conserved.

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mgh = \text{constant}$$

$$v = a\omega \qquad h = a\theta$$

$$\frac{1}{2}ma^2\omega^2 + \frac{1}{2}I\omega^2 + mga\theta = \text{const}$$

unfortunately, not ultimately needed.

<u>Force</u> $\quad T - mg = ma_{cm} = ma\dot{\omega}$

<u>Torque</u> $\quad Ta = N = -I\dot{\omega}$

$$T = -\frac{I\dot{\omega}}{a}$$

Substitute into force

$$-\frac{I\dot{\omega}}{a} - mg = ma\dot{\omega}$$

$$mg = -\left(ma + \frac{I}{a} \right) \dot{\omega}$$

$$\dot{\omega} = \frac{-mg}{ma + \frac{I}{a}}$$

Acceleration

$$a_{cm} = \dot{v} = a\dot{\omega}$$

$$= \frac{-mg\,a}{ma + \frac{I}{a}}$$

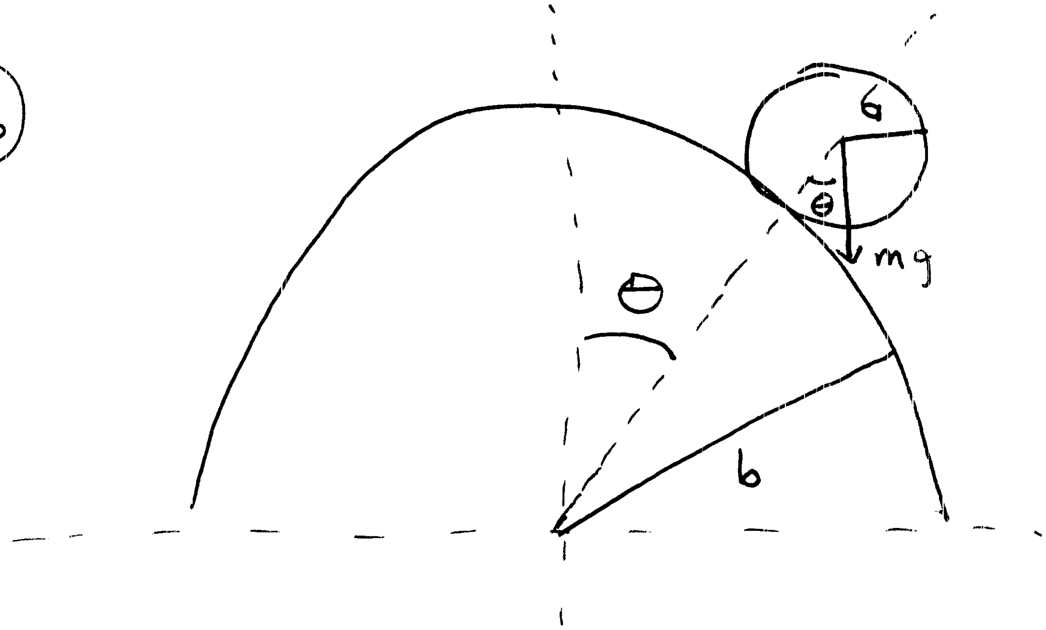
Moment of Inertia Solid Sphere

$$I = \frac{2}{5} ma^2$$

$$a_{cm} = -g \left(\frac{ma}{ma + \frac{2}{5} ma} \right)$$

$$= -\frac{5}{7} g$$

(8.16)



Same problem we did earlier with point mass.

Cylinder leaves surface when normal force is zero.

Since perfectly rough, no slipping. $\Rightarrow$ energy conserved.

__Energy__

$$(\cos\theta)(mg(a+b)) + \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \text{constant}$$

Let $V = 0$ at top.

$$mg(a+b) = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + mg(a+b)\cos\theta$$

__Moment of Inertia Cylinder (wiki)__

$$I = \frac{1}{2}ma^2$$

Condition of rolling

$$v = a\omega$$

Energy

$$mg(a+b) = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}ma^2\right)\omega^2 + mg(a+b)\cos\theta$$

$$= \frac{1}{2}mv^2 + \frac{1}{4}mv^2 + mg(a+b)\cos\theta$$

$$= \frac{3}{4}mv^2 + mg(a+b)\cos\theta$$

Radial Newton II - Inward Positive

$$mg\cos\theta - R = ma_c = \frac{mv^2}{(a+b)}$$

centripetal acceleration

normal force

Cylinder leaves surface when $R=0$

$$g\cos\theta = \frac{v^2}{a+b}$$

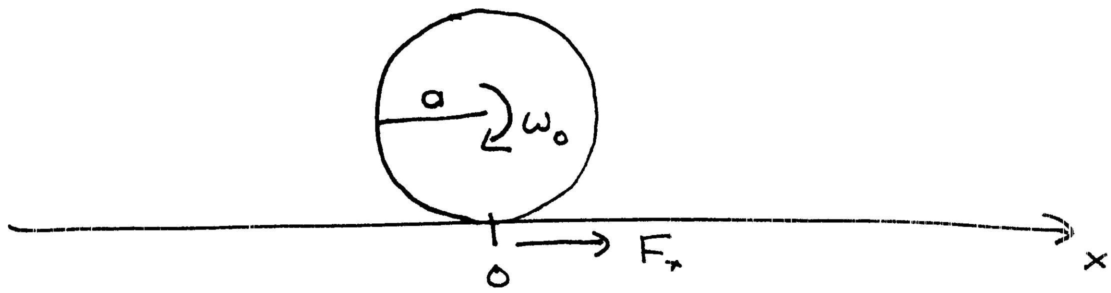
$$mg(a+b) = \tfrac{3}{4} mv^2 + mg(a+b)\cos\theta$$

$$= \frac{3}{4} m \left(g(a+b)\cos\theta \right) + mg(a+b)\cos\theta$$

$$= \left(\frac{3}{4} + 1 \right) mg(a+b)\cos\theta$$

$\cos\theta = 4/7$ $\qquad\qquad$ $\theta = 55^\circ$

8.20



$v_0 = 0$

Slipping occurs, so ball is losing energy. Only friction acts on the ball in the direction of motion

$$F_x = \mu_k m g = m\dot{v} = \text{constant}$$

$$v(t) = \dot{v}\, t = \mu_k g\, t$$

$$x(t) = \frac{1}{2}\dot{v}\, t^2 = \frac{1}{2}\mu_k g\, t^2$$

The frictional force also slows the rotation by exerting a torque

$$N = -F_x a = I\dot{\omega}$$

where I have chosen the initial rotation direction positive.

$$\dot{\omega} = -\frac{\mu_K m g a}{I} = \text{constant}$$

For a solid sphere $I = \frac{2}{5} m a^2$

$$\dot{\omega} = -\frac{5}{2} \frac{g \mu_K}{a}$$

$$\omega(t) = \omega_0 + \dot{\omega} t = \omega_0 - \frac{5}{2} g \frac{\mu_K}{a} t$$

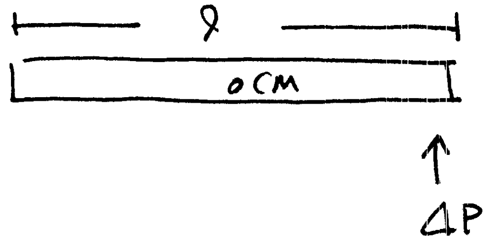
The ball stops sliding when the condition of rolling is met, $v = a\omega$

$$\omega(t) = \omega_0 - \frac{5}{2} g \frac{\mu_K}{a} t = \frac{v}{a} = \frac{\mu_K g t}{a}$$

$$\omega_0 = \left(1 + 5/2\right) \frac{g \mu_K}{a} t$$

$$t = \frac{2}{7} \frac{a \omega_0}{\mu_K g}$$

8.22 I couldn't resist assigning this again.



Since free rotation, must be worked about CM.

$$V_{cm} = \frac{\Delta P}{M}$$

Angular Momentum

$$\Delta L_{cm} = \frac{l}{2}\Delta P = I_{cm}\,\omega$$

$$\omega = \frac{l\,\Delta P}{2I}$$

For a thin rod $\quad I = \cancel{\tfrac{1}{12}\sigma m}\ \frac{1}{12} m l^2$

$$\omega = \frac{12}{2}\frac{\Delta P}{ml} = 6\frac{\Delta P}{ml}$$

The velocity of a point on the rod is

$$v(x) = v_{cm} + x\omega$$

x measured from center

$$v = 0 \implies x = \frac{v_{cm}}{\omega}$$

$$= \frac{\Delta P/m}{6\Delta P/m\ell} = \frac{1}{6}\ell$$