Homework 5

Due Tuesday 2/27/2009

Fowles Problems

1.18

2.12

2.14

3.10

4.20

4.21

Problem E.1 An oscillator is overdamped such that the damping constant is twice the critical damping constant. Write the trajectory of the oscillator in terms of the natural angular frequency. The oscillator is released with amplitude A.

Problem E.2 Consider the force

$$\vec{F} = \frac{\hat{e}_\theta}{r}$$

expressed in spherical coordinates where θ is the angle that $\vec{r}$ makes with the z axis and r is the distance from the origin. Determine whether the force is conservative by evaluating the curl of the force in spherical coordinates.

Problem E.3 An isotropic harmonic oscillator has potential function.

$$V(x, y) = \frac{1}{2}k(x^2 + y^2)$$

where k is the spring constant. The oscillator is confined to the $x - y$ surface. The mass experiencing the restoring force is m. The mass is released from the point $(a, a, 0)$ and brought to a stop at $(-b, -b, 0)$ where a and b are positive constants. The surface has a coefficient of kinetic friction of μ^k. How much work does the force that brings the particle to a stop do?

(1.18) Fly travels in path

$$\vec{r}(t) = b\sin\omega t\,\hat{x} + b\cos\omega t\,\hat{y} + ct^2\,\hat{z}$$

Compute acceleration

$$\vec{v} = \dot{\vec{r}} = \omega b\cos\omega t\,\hat{x} - \omega b\sin\omega t\,\hat{y} + 2ct\,\hat{z}$$

$$\vec{a} = \dot{\vec{v}} = -\omega^2 b\sin\omega t\,\hat{x} - \omega^2 b\cos\omega t\,\hat{y} + 2c\,\hat{z}$$

The magnitude of the acceleration is found by

$$|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{\omega^4 b^2(\sin^2\omega t + \cos^2\omega t) + 4c^2}$$

$$= \sqrt{\omega^4 b^2 + 4c^2}$$

2.12 Bullet Fired Straight Up

Drag Force $\vec{F}_d = -c_2 v|v|$

EOM Upward (Drag downward)

Let z be positive upward

Newton $m\ddot{z} = -mg - c_2 \dot{z}^2$

$$\ddot{z} = -g - k\dot{z}^2 \qquad k \equiv \frac{c_2}{m}$$

Let $v = \dot{z}$

$$\frac{dv}{dt} = -g - kv^2$$

$$\frac{dv}{dt} = \frac{dv}{dz}\frac{dz}{dt} = v\frac{dv}{dz} = -(g + kv^2)$$

Assume fired from $z=0$ at $t=0$ with velocity v_0.

$$\int_0^v \frac{v\,dv}{g+kv^2} = -\int_0^z dz$$

U substitution

$$u = g + kv^2 \qquad du = 2kv\,dv$$

$$\int \frac{v\,dv}{g+kv^2} = \frac{1}{2k}\int \frac{du}{u}$$

$$= \frac{1}{2k} \ln u + C$$

$$= \frac{1}{2k} \ln (g + kv^2) + C$$

Apply Limits

$$\int_{v_0}^{v} \frac{v\,dv}{g+kv^2} = \frac{1}{2k}\left[\ln(g+kv^2) - \ln(g+kv_0^2)\right]$$

$$= \frac{1}{2k} \ln\left(\frac{g+kv^2}{g+kv_0^2}\right)$$

$$= -z$$

$$\ln\left(\frac{g+kv^2}{g+kv_0^2}\right) = -2kz$$

$$g + kv^2 = (g + kv_0^2)e^{-2kz}$$

$$v^2 = \frac{1}{k}\left[(g + kv_0^2)e^{-2kz} - g\right]$$

$$= A e^{-2kz} - \frac{g}{k}$$

$$A = \frac{g}{k} + v_0^2$$

Note – v_{max} could be found by setting $v^2 = 0$.

Downward motion

Let the ground be $x=0$, upward be positive, and the initial height be z_0.

EOM $\quad m\ddot{z} = -mg + c_2\dot{z}^2$

$$\ddot{z} = -g + k\dot{z}^2 \qquad k = \frac{c_2}{m}$$

<u>Same trick</u> $v = \dot{z}$

$$\frac{dv}{dt} = \frac{dv}{dz}\frac{dz}{dt} = v\frac{dv}{dz}$$

$$\ddot{z} = v\frac{dv}{dz} = -g + kv^2$$

$$\int_0^v \frac{v\,dv}{-g+kv^2} = \int_{z_0}^{z} dz = z - z_0$$

$$u = -g + kv^2 \qquad du = 2kv^2$$

$$\int_0^v \frac{v\,dv}{-g+kv^2} = \frac{1}{2k}\int_{-g}^{-g+kv^2} \frac{du}{u} = \frac{1}{2k}\ln\left(\frac{-g+kv^2}{-g}\right)$$

$$= z - z_0$$

$$\ln\left(\frac{-g+kv^2}{-g}\right) = 2k(z-z_0)$$

$$\frac{-g+kv^2}{-g} = e^{2k(z-z_0)} = 1 - \frac{k}{g}v^2$$

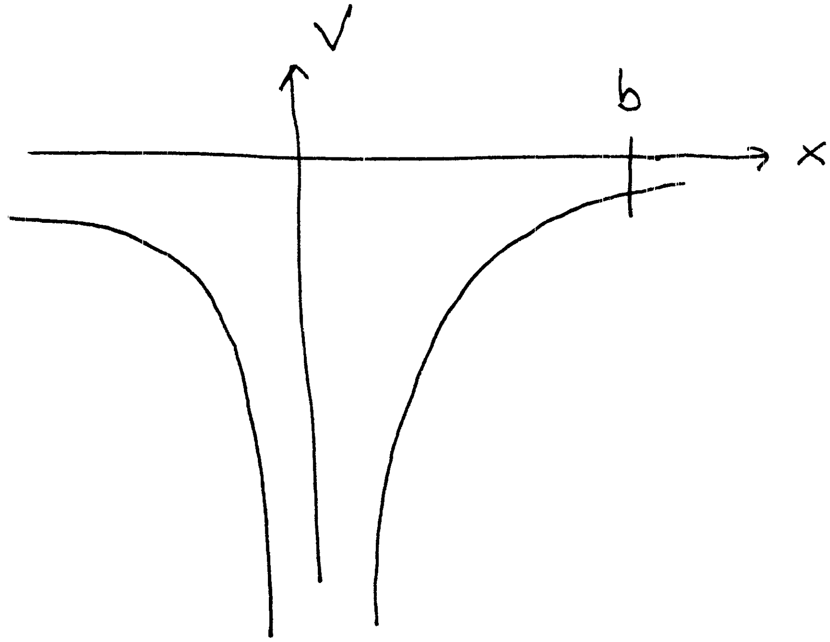
$$1 - e^{2k(z-z_0)} = \frac{k}{g} v^2$$

$$v^2 = \frac{g}{k} - \frac{g}{k} e^{2k(z-z_0)}$$

$$= \frac{g}{k} - B e^{2kz}$$

$$B \equiv \frac{g}{k} e^{-2kz_0}$$

2.14



$$F(x) = -\frac{k}{x^2} \qquad V(x) = -\frac{k}{x}$$

$$F = -\frac{dV}{dx} \qquad \text{Let } V(\infty) = 0$$

<u>Conserve Energy</u>

$$U = U_0 = -\frac{k}{b} = U(t) = -\frac{k}{x} + \frac{1}{2} m v^2$$

$$\frac{k}{x} - \frac{k}{b} = \frac{1}{2} m v^2$$

$$v^2 = 2\frac{k}{m}\left(\frac{1}{x} - \frac{1}{b}\right)$$

$$v = -\sqrt{\Gamma\left(\frac{1}{x} - \frac{1}{b}\right)} \qquad \Gamma \equiv 2\frac{k}{m}$$

(Note, negative root because going left)

Maple - But simplify first

$$v = -\sqrt{\Gamma\left(\frac{b-x}{bx}\right)} = \frac{dx}{dt}$$

$$= -\sqrt{\frac{\Gamma}{b}}\sqrt{\frac{b-x}{x}} = \frac{dx}{dt}$$

$$\int_0^t dt = -\int_b^0 \frac{dx}{\sqrt{\frac{\Gamma}{b}}\sqrt{\frac{b-x}{x}}}$$

$$\sqrt{\frac{\Gamma}{b}}\, t = -\int_b^0 \sqrt{\frac{x}{b-x}}\, dx$$

Using Maple with assume $(b>0)$ yields

$$\int_b^0 \sqrt{\frac{x}{b-x}}\, dx = -\frac{b\pi}{2}$$

$$t = \sqrt{\frac{b}{\Gamma}}\,\frac{b\pi}{2} = \frac{\pi b}{2}\left(\frac{b}{2k/m}\right)^{1/2}$$

$$t = \pi \left(\frac{b^3 m}{8k} \right)^{1/2}$$

Note, one can approach this by hand
using $u = \sin^2 \theta$

3.10

The amplitude of driven oscillations is given by (3.6.9)

$$A(\omega) = \frac{F_0/m}{\left[(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2\right]^{1/2}}$$

$\omega_0 = \sqrt{\frac{k}{m}}$ the natural frequency

$$= \sqrt{\frac{250\,N/m}{10\,kg}} = 5\,s^{-1}$$

$\omega \equiv$ Driving frequency

$$\gamma = \frac{c}{2m} = \frac{60\,kg/s}{2(10\,kg)} = 3\,s^{-1}$$

$\equiv$ Damping factor

The maximum amplitude happens at the resonant frequency where

$$\left.\frac{dA}{d\omega}\right|_{\omega_r} = 0$$

The text solves this equation to give

$$\omega_r^2 = \omega_0^2 - 2\gamma^2$$

$$= (5s^{-1})^2 - 2(3s^{-1})^2$$

$$= (25-18)s^{-2} = 7s^{-2}$$

$$\boxed{\omega_r = \sqrt{7}\ s^{-1}}$$

(b) $$A(\omega_r) = \frac{48N/10kg}{\left(\left((5s^{-1})^2 - (\sqrt{7}\ s^{-1})^2\right)^2 + 4(3s^{-1})^2(\sqrt{7}s^{-1})^2\right)^{1/2}}$$

~~48N~~

$$A(\omega_r) = \frac{48N/10\,kg}{\left((18\,s^{-2})^2 + 252\,s^{-4}\right)^{1/2}}$$

$$= \frac{48\,N/10\,kg}{24\,s^{-2}} = 0.2\,m$$

(c) Phase angle (3.6.8)

$$\tan\phi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}$$

$$= \frac{2(3s^{-1})(\sqrt{7}s^{-1})}{(5s^{-1})^2 - (\sqrt{7}\,s^{-1})^2}$$

$$\tan\phi = \frac{6\sqrt{7}}{18} = \frac{\sqrt{7}}{3}$$

4.20 Same solution as in lecture with a twist

$$\vec{E} = E\hat{y} \qquad \vec{B} = B\hat{z} \qquad q = -e$$

$$\vec{r}_0 = 0 \qquad \vec{V}_0 = V_0\hat{x}$$

<u>Lorentz Force</u> $\qquad \vec{F} = q\vec{E} + q\,\vec{v}\times\vec{B}$

<u>Take cross-product</u>

$$\cancel{q}\;\vec{v}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & 0 & B \end{vmatrix}$$

$$= \dot{y}B\hat{x} - \dot{x}B\hat{y}$$

$$\vec{F} = -eE\hat{y} - eB\dot{y}\hat{x} + eB\dot{x}\hat{y}$$

<u>EOM</u>

$$m\ddot{x} = -eB\dot{y}$$

$$m\ddot{y} = -eE + eB\dot{x}$$

$$\ddot{x} = -\Gamma \dot{y} \qquad \Gamma \equiv eB/m$$

$$\ddot{y} = -\gamma + \Gamma \dot{x} \qquad \gamma \equiv eE/m$$

Integrate each once

$$\dot{x} = -\Gamma y + C_1$$

Initial condition $\quad x(0) = 0 \quad \dot{x}(0) = v_0$
$\quad y(0) = 0$

$$\dot{x}(0) = v_0 = -\Gamma \cdot 0 + C_1 = C_1$$

$$C_1 = v_0$$

$$\dot{x} = -\Gamma y + v_0$$

$$\int \frac{d\dot{y}}{dt} dt = -\int \gamma dt + \Gamma \int \frac{dx}{dt} dt$$

$$\dot{y} = -\gamma t + \Gamma x + C_2$$

<u>Initial Condition</u> $\quad y(0) = 0 \quad x(0) = 0 \quad \dot{y}(0) = 0$

$$\Rightarrow C_2 = 0$$

Our two integrated equations are

$$\dot{x} = -\Gamma y + v_0$$

$$\dot{y} = -\gamma t + \Gamma x$$

substitute into original equations

$$\ddot{x} = -\Gamma \dot{y}$$

$$\ddot{y} = -\gamma + \Gamma \dot{x}$$

yielding

$$\ddot{y} = -\gamma + \Gamma(-\Gamma y + v_0)$$

$$= -\gamma - \Gamma^2 y + \Gamma v_0$$

$$\ddot{y} + \Gamma^2 y = \Gamma v_0 - \gamma$$

$$\ddot{x} = -\Gamma(-\gamma t + \Gamma x)$$

$$\ddot{x} = -\Gamma^2 x + \gamma \Gamma t$$

$$\ddot{x} + \Gamma^2 x = \gamma \Gamma t$$

Solve $\ddot{y} + \Gamma^2 y = \Gamma v_0 - \gamma$

$$y = y_h(t) + y_p(t)$$

Homogeneous Solution

$$y_h(t) = A\cos\Gamma t + B\sin\Gamma t$$

Particular Solution (Guess)

$$y_p(t) = C_y \qquad \ddot{y}_p = 0$$

$$0 + \Gamma^2 C_y = \Gamma v_0 - \gamma$$

$$\boxed{C_y = \frac{\Gamma v_0 - \gamma}{\Gamma^2}}$$

$$y(t) = y_h + y_p = A\cos\Gamma t + B\sin\Gamma t + C_y$$

Initial Conditions $\quad y(0) = 0 \quad \dot{y}(0) = 0$

$$y(0) = A + C_y = 0 \qquad A = -C_y$$

$$\dot{y}(0) = 0 = B\Gamma \implies B = 0$$

$$y(t) = C_y(1 - \cos\Gamma t)$$

Solve $\ddot{x} + \Gamma^2 x = \gamma \Gamma t$

$$x(t) = x_h(t) + x_p(t)$$

Homogeneous Solution

$$x_h = A\cos\Gamma t + B\sin\Gamma t \quad \text{as before}$$

Particular Solution (Guess)

$$x_p(t) = C_x t + D_x$$

$$\ddot{x}_p = 0$$

$$0 + \Gamma^2(C_x t + D_x) = \gamma \Gamma t$$

$$D_x = 0 \qquad C_x = \frac{\gamma}{\Gamma}$$

$$x(t) = A\cos\Gamma t + B\sin\Gamma t + \frac{\gamma}{\Gamma} t$$

Initial Conditions $x(0) = 0$ $\dot{x}(0) = v_0$

$$x(0) = A = 0 \implies A = 0$$

$$\dot{x}(0) = v_0 = B\Gamma + \frac{\gamma}{\Gamma}$$

$$B = \frac{v_0 - \frac{\gamma}{\Gamma}}{\Gamma} = \frac{\Gamma v_0 - \gamma}{\Gamma^2} = C_y$$

$$x(t) = C_y \sin \Gamma t + \frac{\gamma}{\Gamma} t$$

So to map onto the text's solution

$$a = C_y \qquad b = \gamma / \Gamma$$

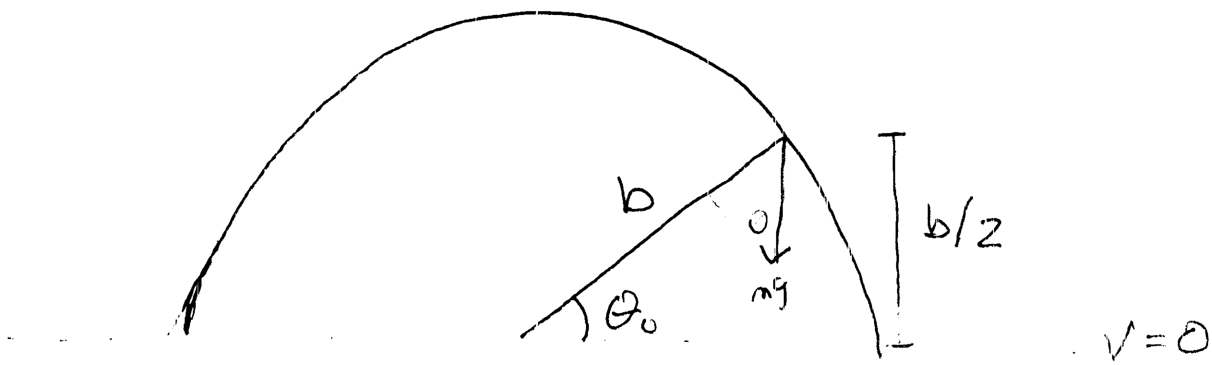
Let's work out what these constants are in terms of the fields.

$$a = C_y = \frac{\Gamma v_0 - \gamma}{\Gamma^2}$$

$$= \frac{\frac{eB}{m} v_0 - \frac{eE}{m}}{\left(\frac{eB}{m}\right)^2} = \boxed{\frac{m}{Be}\left(v_0 - \frac{E}{B}\right)}$$

$$\boxed{b = \frac{E}{B}}$$

4.21



<u>Initial angle</u>

$$\cancel{\cos}\ \sin\theta_0 = \frac{b/2}{b}$$

$$\theta_0 = 30^\circ$$

<u>Conserve Energy</u>

$$U_{sy.} = \frac{mgb}{2} = \frac{1}{2}mv^2 + U(\theta)$$

$$U(\theta) = mgb\sin\theta$$

$$v^2 = \frac{2}{m}\left(\frac{mgb}{2} - mgb\sin\theta\right)$$

$$= gb(1 - 2\sin\theta)$$

Newton II ~~For~~ $\perp$ to surface

$$m a_c = R - mg \sin\theta$$

centripetal acceleration

component of force $\perp$ surface

$$-\frac{mv^2}{b} = R - mg \sin\theta$$

Particle comes of surface when normal force is zero, at

$$-\frac{mv^2}{b} = -mg \sin\theta$$

$$v^2 = gb \sin\theta$$

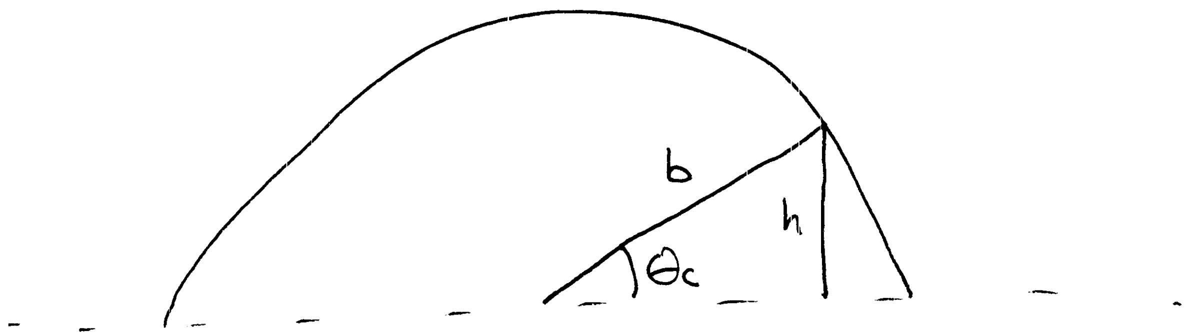
Substitute expression derived from energy equation

$$v^2 = gb(1 - 2\sin\theta$$

$$= gb \sin\theta$$

$$1 = 3\sin\theta$$

$\sin\theta_c = 1/3$ when bead comes off sphere.



This is at a height of $h = b/3$

E1 Damping constant = Twice Critically Damped

Critical damping factor $\gamma_c = \omega_0$

$\gamma = 2\omega_0$

Since $\gamma > \gamma_c$, system overdamped

$$q = \sqrt{\gamma^2 - \omega_0^2} = \sqrt{3}\,\omega_0$$

Solutions

$$x(t) = A' e^{-(\gamma+q)t} + B e^{-(\gamma-q)t}$$

$$x(t) = A' e^{-(2+\sqrt{3})\omega_0 t} + B e^{-(2-\sqrt{3})\omega_0 t}$$

Initial Conditions $\quad x(0) = A \quad \dot{x}(0) = 0$

$$x(0) = A' + B = A$$

$$\cancel{\dot{x}(0) = -A'(2+\sqrt{3})\omega_0 e^{-(2+\sqrt{3})\omega_0 t}}$$

$$\dot{x}(0) = -A'(2+\sqrt{3})\omega_0 - B(2-\sqrt{3})\omega_0 = 0$$

$$A' = A - B \qquad A' = -\frac{(2-\sqrt{3})}{2+\sqrt{3}} B$$

$$-\frac{(2-\sqrt{3})}{2+\sqrt{3}} B = A - B$$

$$B\left(1 - \frac{(2-\sqrt{3})}{2+\sqrt{3}}\right) = A$$

$$B\left(\frac{2+\sqrt{3} - 2 + \sqrt{3}}{2+\sqrt{3}}\right) = A$$

$$B = \frac{2+\sqrt{3}}{2\sqrt{3}} A$$

$$A' = -\left(\frac{2-\sqrt{3}}{2+\sqrt{3}}\right)\frac{2+\sqrt{3}}{2\sqrt{3}} A$$

$$= \frac{-(2-\sqrt{3})}{2\sqrt{3}} A$$

E2

$$\vec{F} = \frac{\hat{e}_\theta}{r} \qquad \text{spherical}$$

$$\nabla \times \vec{F} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\, F_\phi) - \frac{\partial F_\theta}{\partial\phi}\right]\hat{r}$$

$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}(r F_\phi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r F_\theta) - \frac{\partial F_r}{\partial\theta}\right]\hat{\phi}$$

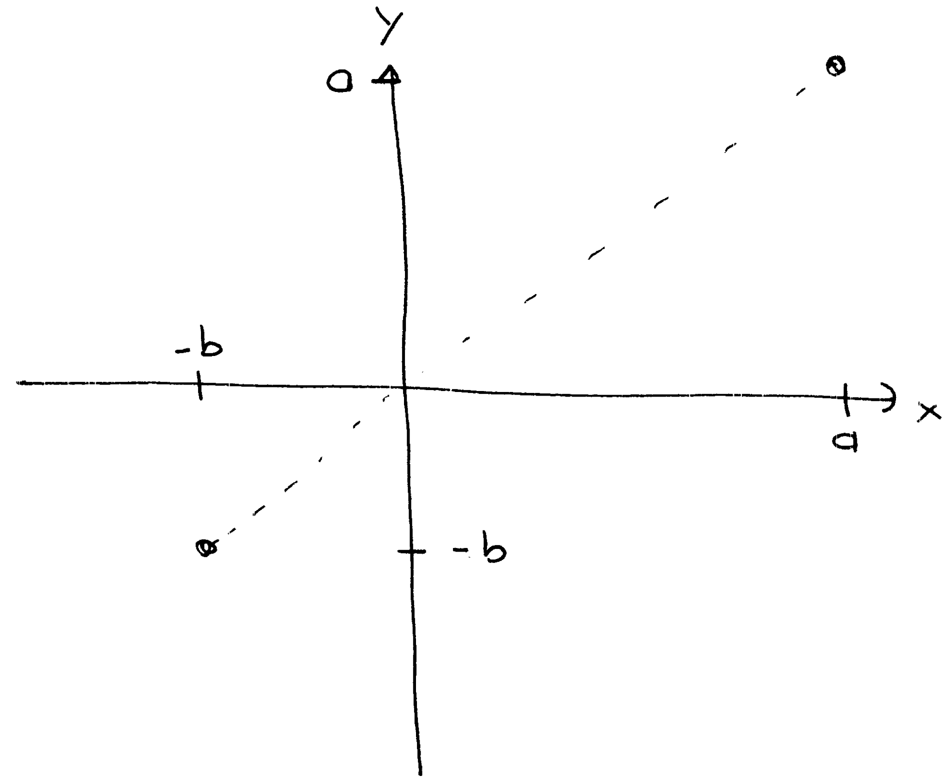
$$\vec{F} = 0\,\hat{r} + 0\,\hat{\phi} + \frac{1}{r}\hat{\theta}$$

$$\nabla \times \vec{F} = \frac{1}{r\sin\theta}\left(-\frac{\partial}{\partial\phi}\frac{1}{r}\right)\hat{r}$$

$$+ \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\cdot\frac{1}{r}\right)\right)\hat{\phi}$$

$$= 0$$ so the force is conservative

E3



The energy dissipated alone the trajectory is

$$E_{diss} = \mu^{K} N d = mg\mu^{K} d$$

where $d = \sqrt{2}(a+b)$ the length of the path

The work done to stop the particle is the remaining potential energy minus the dissipated energy.

$$W = V(a,a,0) - V(b,b,0) - E_{diss}$$

$$= K(a^2-b^2) - mg\mu^{K}\sqrt{2}(a+b)$$