Homework 6

Due Tuesday 11/3/2009 at 5:30pm in my box in physics.

Fowles Problems

10.6

10.10

10.12

10.19

10.26

10.27 Part (a) only

10.3 Find the acceleration of a solid uniform sphere rolling down a perfectly rough, fixed inclined plane. Compare with the result derived earlier in Section 8.6.

10.4 Two blocks of equal mass m are connected by a flexible cord. One block is placed on a smooth horizontal table, the other block hangs over the edge. Find the acceleration of the blocks and cord assuming **(a)** the mass of the cord is negligible and **(b)** the cord is heavy, of mass m'.

10.5 Set up the equations of motion of a "double-double" Atwood machine consisting of one Atwood machine (with masses m_1 and m_2) connected by means of a light cord passing over a pulley to a second Atwood machine with masses m_3 and m_4. Ignore the masses of all pulleys. Find the accelerations for the case $m_1 = m$, $m_2 = 4m$, $m_3 = 2m$, and $m_4 = m$.

10.6 A ball of mass m rolls down a movable wedge of mass M. The angle of the wedge is θ, and it is free to slide on a smooth horizontal surface. The contact between the ball and the wedge is perfectly rough. Find the acceleration of the wedge.

10.7 A particle slides on a smooth inclined plane whose inclination θ is increasing at a constant rate ω. If $\theta = 0$ at time $t = 0$, at which time the particle starts from rest, find the subsequent motion of the particle.

10.8 Show that Lagrange's method automatically yields the correct equations of motion for a particle moving in a plane in a rotating coordinate system Oxy. (*Hint:* $T = \frac{1}{2} m\mathbf{v}\cdot\mathbf{v}$, *where* $\mathbf{v} = \mathbf{i}(\dot{x} - \omega y) + \mathbf{j}(\dot{y} + \omega x)$, and $F_x = -\partial V/\partial x$, $F_y = -\partial V/\partial y$.)

10.9 Repeat Problem 10.8 for motion in three dimensions.

10.10 Find the differential equations of motion for an "elastic pendulum": a particle of mass m attached to an elastic string of stiffness K and unstretched length l_0. Assume that the motion takes place in a vertical plane.

10.11 A particle is free to slide along a smooth cycloidal trough whose surface is given by the parametric equations

$$x = \frac{a}{4}(2\theta + \sin 2\theta)$$

$$y = \frac{a}{4}(1 - \cos 2\theta)$$

where $0 \le \theta \le \pi$ and a is a constant. Find the Lagrangian function and the equation of motion of the particle.

10.12 A simple pendulum of length l and mass m is suspended from a point on the circumference of a thin massless disc of radius a that rotates with a constant angular velocity ω about its central axis as shown in Figure P10.12. Find the equation of motion of the mass m.

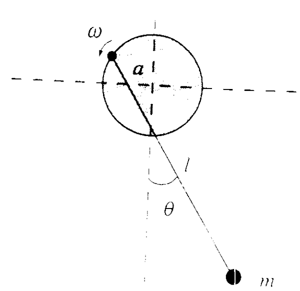


Figure P10.12

10.13 A bead of mass m is constrained to slide along a thin, circular hoop of radius l that rotates with constant angular velocity ω in a horizontal plane about a point on its rim as shown in Figure P10.13.

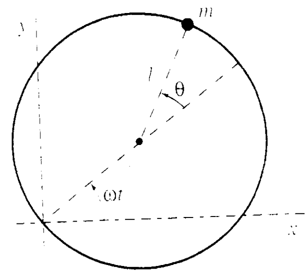


Figure P10.13

(a) Find Lagrange's equation of motion for the bead.

(b) Show that the bead oscillates like a pendulum about the point on the rim diametrically opposite the point about which the hoop rotates.

(c) What is the effective "length" of this "pendulum"?

10.14 The point of support of a simple pendulum is being elevated at a constant acceleration a, so that the height of the support is $\frac{1}{2}at^2$, and its vertical velocity is at. Find the differential equation of motion for small oscillations of the pendulum by Lagrange's method. Show that the period of the pendulum is $2\pi[l/(g + a)]^{1/2}$, where l is the length of the pendulum.

10.15 Work Problem 8.12 by using the method of Lagrange multipliers. **(a)** Show that the acceleration of the ball is $\frac{5}{7}g$. **(b)** Find the tension in the string.

10.16 A heavy elastic spring of uniform stiffness and density supports a block of mass m. If m' is the mass of the spring and k its stiffness, show that the period of vertical oscillations is

$$2\pi\sqrt{\frac{m + (m'/3)}{k}}$$

This problem shows the effect of the mass of the spring on the period of oscillation. *Hint: To set up the Lagrangian function for the system, assume that the velocity of any part of the spring is proportional to its distance from the point of suspension.*

10.17 Use the method of Lagrange multipliers to find the tensions in the two strings of the double Atwood machine of Example 10.5.4.

10.18 A smooth rod of length l rotates in a plane with a constant angular velocity ω about an axis fixed at one end of the rod and perpendicular to the plane of rotation. A bead of mass m is initially positioned at the stationary end of the rod and given a slight push such that its initial speed directed along the rod is ωl.

(a) Find the time it takes the bead to reach the other end of the rod.

(b) Use the method of Lagrange multipliers to find the reaction force **F** that the rod exerts on the bead.

10.19 A particle of mass m perched on top of a smooth hemisphere of radius a is disturbed ever so slightly, so that it begins to slide down the side. Find the normal force of constraint exerted by the hemisphere on the particle and the angle relative to the vertical at which it leaves the hemisphere. Use the method of Lagrange multipliers.

yields the correct equation of motion where $\mathbf{B} = \nabla \times \mathbf{A}$. The quantity $\mathbf{A}$ is called the *vector potential. Hint: In this problem it will be necessary to employ the general formula* $df(x, y, z)/dt = \dot{x}\,\partial f/\partial x + \dot{y}\,\partial f/\partial y + \dot{z}\,\partial f/\partial z$. *Thus, for the part involving* $\mathbf{v} \cdot \mathbf{A}$, *we have*

$$\frac{d}{dt}\left[\frac{\partial(\mathbf{v} \cdot \mathbf{A})}{\partial \dot{x}}\right] = \frac{d}{dt}\left[\frac{\partial}{\partial \dot{x}}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z)\right] = \frac{d}{dt}(A_x)$$

$$= \dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_x}{\partial y} + \dot{z}\frac{\partial A_x}{\partial z}$$

and similarly for the other derivatives.

10.26 Write the Hamiltonian function and find Hamilton's canonical equations for the three-dimensional motion of a projectile in a uniform gravitational field with no air resistance. Show that these equations lead to the same equations of motion as found in Section 4.3

10.27 Find Hamilton's canonical equations for

(a) A simple pendulum

(b) A simple Atwood machine

(c) A particle sliding down a smooth inclined plane

10.28 A particle of mass m is subject to a central, attractive force given by

$$\mathbf{F}(r, t) = -\mathbf{e}_r \frac{k}{r^2} \exp(-\beta t)$$

where k and β are positive constants, t is the time, and r is distance to the center of force. **(a)** Find the Hamiltonian function for the particle. **(b)** Compare the Hamiltonian to the total energy of the particle. **(c)** Is the energy of the particle conserved? Discuss.

10.29 Two particles whose masses are m_1 and m_2 are connected by a massless spring of unstressed length l and spring constant k. The system is free to rotate and vibrate on top of a smooth horizontal plane that serves as its support. **(a)** Find the Hamiltonian of the system. **(b)** Find Hamilton's equations of motion. **(c)** What generalized momenta, if any, are conserved?

10.30 As we know, the kinetic energy of a particle in one-dimensional motion is $\frac{1}{2}m\dot{x}^2$. If the potential energy is proportional to x^2, say $\frac{1}{2}kx^2$, show by direct application of Hamilton's variational principle, $\delta \int L\, dt = 0$, that the equation of the simple harmonic oscillator is obtained.

COMPUTER PROBLEMS

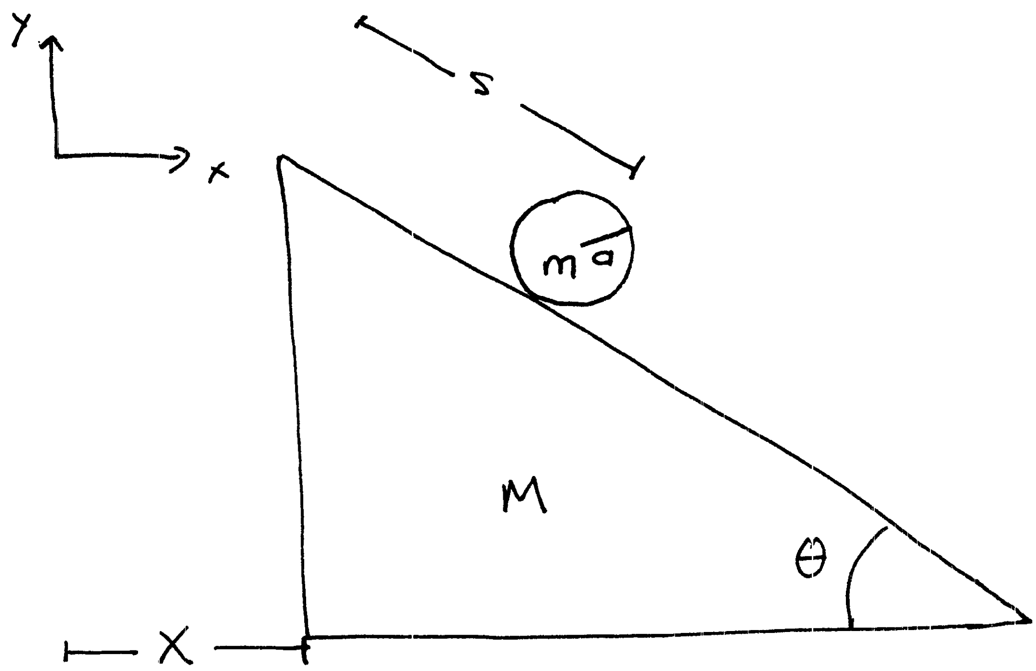
C 10.1 Assume that the spherical pendulum discussed in Section 10.6 is set into motion with the following initial conditions: $\phi_0 = 0$ rad, $\dot{\phi}_0 = 10.57$ rad/s, $\theta_0 = \pi/4$ rad, and $\dot{\theta}_0 = 0$ rad/s. Let the length of the pendulum be 0.284 m.

(a) Calculate θ_1 and θ_2, the polar angular limits of the motion.

$$\frac{1}{2}\dot{\theta}^2 = -U(\theta) - C = 0$$

Hint: Solve the equation numerically for the condition of $\dot{\theta} = 0$

10.6



Let the location of the wedge be given by X, the location of the ball by s. The moment of inertia of the ball is $I = \frac{2}{5} m a^2$.

Let (x_{cm}, y_{cm}) be the location of the center of the ball.

$$V = - m g s \sin\theta$$

$$x_{cm} = X + s\cos\theta \qquad \cancel{y_{cm} = X}$$

$$y_{cm} = - s \sin\theta$$

$$\dot{x}_{cm} = \dot{X} + \dot{s}\cos\theta \qquad \dot{y}_{cm} = -\dot{s}\sin\theta$$

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{x}_{cm}^2 + \dot{y}_{cm}^2\right) + \cancel{\frac{1}{2}I\dot{\theta}^2} + \frac{1}{2}I\omega^2$$

Condition of rolling without slipping $\quad \omega a = v = \dot{s}$

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\left(\dot{X} + \dot{s}\cos\theta\right)^2 + \left(\dot{s}\sin\theta\right)^2\right)$$
$$+ \frac{1}{2}I\left(\frac{\dot{s}}{a}\right)^2$$

$$= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{X}^2 + 2\dot{s}\dot{X}\cos\theta + \dot{s}^2\cos^2\theta + \dot{s}^2\sin^2\theta\right)$$
$$+ \frac{1}{2}\frac{I}{a^2}\dot{s}^2$$

$$= \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}\left(m + \frac{I}{a^2}\right)\dot{s}^2 + m\dot{s}\dot{X}\cos\theta$$

$$L = T - V = \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}\left(m + \frac{I}{a^2}\right)\dot{s}^2$$
$$+ m\dot{s}\dot{X}\cos\theta + mgs\sin\theta$$

x EOM

x does not appear in L $\Rightarrow$ x is an ignorable coordinate $\Rightarrow$ $\dot{x}$ = constant

$$\frac{\partial L}{\partial x} = 0 \qquad \frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x} + m\dot{s}\cos\theta$$

$$0 = \frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = (M+m)\ddot{x} + m\ddot{s}\cos\theta = 0$$

$$\ddot{x} = -\frac{m\cos\theta}{M+m}\ddot{s}$$

s EOM

$$\frac{\partial L}{\partial s} = mg\sin\theta$$

$$\frac{\partial L}{\partial \dot{s}} = \left(m + \frac{I}{a^2}\right)\dot{s} + m\dot{x}\cos\theta$$

$$0 = \frac{\partial L}{\partial s} - \frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = mg\sin\theta - \left(m + \frac{I}{a^2}\right)\ddot{s} - m\ddot{x}\cos\theta$$

$$\cancel{-\left(m + \frac{I}{a^2}\right)}$$

Sub X EOM

$$mg\sin\theta - m\cos\theta\left(\frac{-m\cos\theta}{M+m}\right)\ddot{s} - \left(m+\frac{I}{a^2}\right)\ddot{s} = 0$$

$$g\sin\theta = \left(1+\frac{I}{ma^2} - \cos^2\theta\left(\frac{m}{M+m}\right)\right)\ddot{s}$$

$$\ddot{s} = A = \frac{g\sin\theta}{1+\frac{I}{ma^2} - \cos^2\theta\left(\frac{m}{M+m}\right)}$$

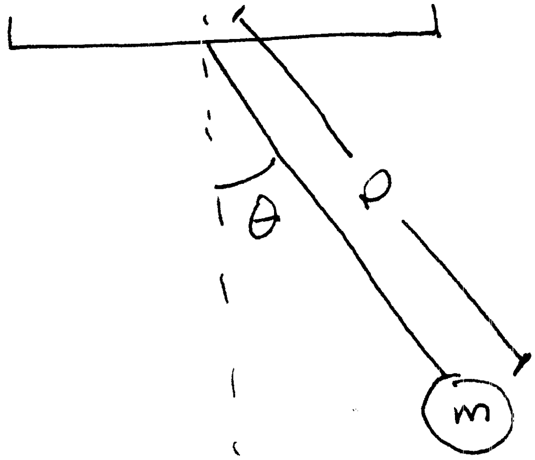
If $I = \frac{2}{5}ma^2$,

$$\ddot{s} = A = \frac{g\sin\theta\,(M+m)}{\frac{7}{5}(M+m) - m\cos^2\theta}$$

So $\ddot{x} = \frac{-m\cos\theta}{M+m}\ddot{s} = \frac{-m\cos\theta}{M+m}\cdot\frac{g\sin\theta\,(M+m)}{\frac{7}{5}(M+m) - m\cos^2\theta}$

$$= \frac{-mg\sin\theta\cos\theta}{\frac{7}{5}(M+m) - m\cos^2\theta}$$

10.10



$$V = mg(\ell_0 - \ell\cos\theta) + \tfrac{1}{2}k(\ell - \ell_0)^2$$

Let $V(0)$ be hanging straight down with unstretched length.

$$x = \ell\sin\theta \qquad y = (\ell_0 - \ell\cos\theta)$$

$$\dot{x} = \dot{\ell}\sin\theta + \ell\cos\theta\,\dot{\theta}$$

$$\dot{y} = -\dot{\ell}\cos\theta + \ell\sin\theta\,\dot{\theta}$$

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \tfrac{1}{2}m\left(\dot{\ell}^2 + (\ell\dot{\theta})^2\right)$$

$$L = T - V = \tfrac{1}{2}m\dot{\ell}^2 + \tfrac{1}{2}m\ell^2\dot{\theta}^2$$
$$- mg(\ell_0 - \ell\cos\theta) - \tfrac{1}{2}k(\ell - \ell_0)^2$$

ℓ EOM

$$\frac{\partial L}{\partial \ell} = m\ell\dot{\theta}^2 + mg\cos\theta - k(\ell-\ell_0)$$

$$\frac{\partial L}{\partial \dot{\ell}} = m\dot{\ell}$$

$$0 = \frac{\partial L}{\partial \ell} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\ell}} = m\ell\dot{\theta}^2 + mg\cos\theta - k(\ell-\ell_0) - m\ddot{\ell} = 0$$

$$\ddot{\ell} = -\frac{k}{m}(\ell-\ell_0) + \ell\dot{\theta}^2 + g\cos\theta$$

θ EOM

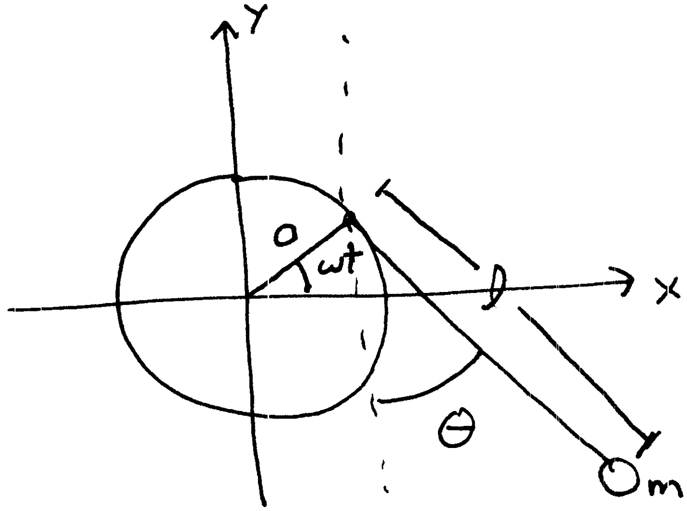
$$\frac{\partial L}{\partial \theta} = -mg\ell\sin\theta \qquad \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m\ell^2\ddot{\theta} + 2m\ell\dot{\theta}\dot{\ell}$$

$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -mg\ell\sin\theta - m\ell^2\ddot{\theta} - 2m\ell\dot{\theta}\dot{\ell}$$

$$\ddot{\theta} + 2\dot{\theta}\frac{\dot{\ell}}{\ell} + \frac{g}{\ell}\sin\theta = 0$$

10.12



Coordinates of mass

$$x(t) = a\cos\omega t + l\sin\theta$$

$$\dot{x}(t) = -a\omega\sin\omega t + l\cos\theta\,\dot{\theta}$$

$$y(t) = a\sin\omega t - l\cos\theta$$

$$\dot{y}(t) = +a\omega\cos\omega t + l\sin\theta\,\dot{\theta}$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) = \frac{1}{2}m\left(-a\omega\sin\omega t + l\cos\theta\,\dot{\theta}\right)^2$$

$$+\frac{1}{2}m\left(a\omega\cos\omega t + l\sin\theta\,\dot{\theta}\right)^2$$

$$= \frac{1}{2}ma^2\omega^2\left(\sin^2\omega t + \cos^2\omega t\right) + \frac{1}{2}ml^2\dot{\theta}^2\left(\cos^2\theta + \sin^2\theta\right)$$

$$+\frac{2}{2}ma\omega l\left(\sin\theta\cos\omega t - \cos\theta\sin\omega t\right)\dot{\theta}$$

Trig Identity (schaums)

$$\cos\omega t \sin\theta - \sin\omega t \cos\theta = \sin(\theta - \omega t)$$

$$T = \frac{1}{2} m a^2 \omega^2 + \frac{1}{2} m l^2 \dot{\theta}^2 + m a \omega l \sin(\theta - \omega t)\dot{\theta}$$

$$L = T - V = \frac{1}{2} m a^2 \omega^2 + \frac{1}{2} m l^2 \dot{\theta}^2 + m a \omega l \sin(\theta - \omega t)\dot{\theta}$$
$$- mg(a \sin\omega t - l\cos\theta)$$

$$\frac{\partial L}{\partial \theta} = m a \omega l \cos(\theta - \omega t)\dot{\theta} - mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} + m a \omega l \sin(\theta - \omega t)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m l^2 \ddot{\theta} + m a \omega l \cos(\theta - \omega t)(\dot{\theta} - \omega)$$

EOM

$$\cancel{0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = m l^2 \ddot{\theta}}$$

EOM

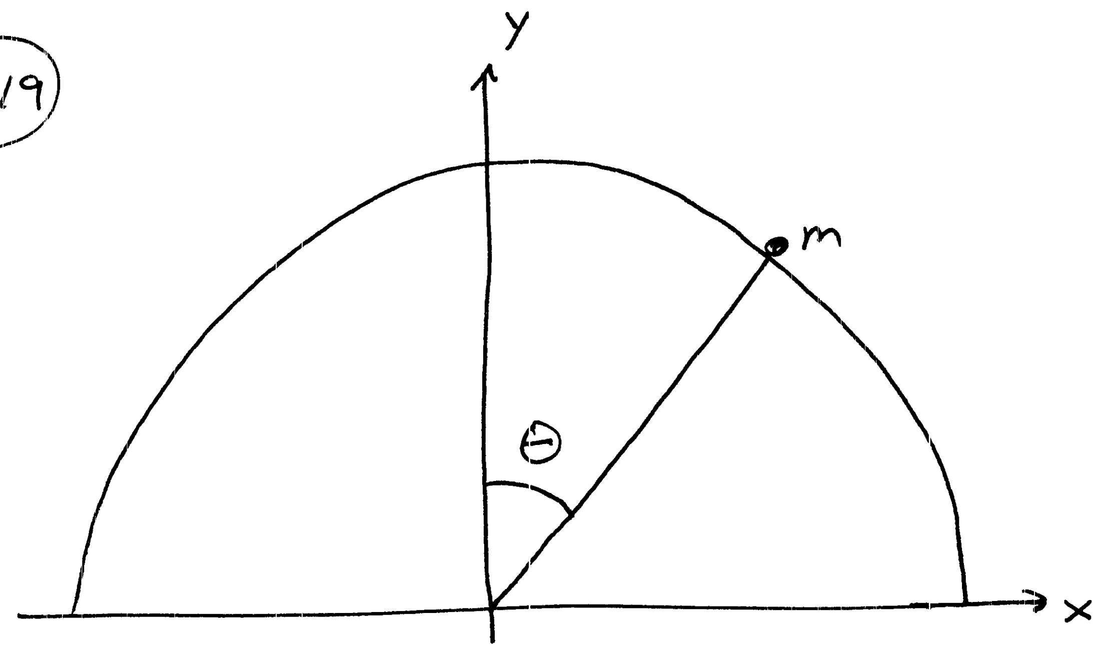
$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$= ma\omega l \cos(\theta - \omega t)\,\dot{\theta} - mgl\sin\theta$$

$$- ml^2\ddot{\theta} - ma\omega l\cos(\theta-\omega t)(\dot{\theta}-\omega) = 0$$

$$\ddot{\theta} - \frac{a\omega^2}{l}\cos(\theta - \omega t) + \frac{g}{l}\sin\theta = 0$$

10.19



Constraint $\quad f = r - a = 0$

Write EOM without using constraint to reduce coordinates.

$$T = \frac{1}{2} m \left(\dot{r}^2 + (r\dot{\theta})^2 \right)$$

$$V = mgr\cos\theta$$

$$L = T - V = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - mgr\cos\theta$$

Use Lagrange Multipliers

$$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda \frac{\partial f}{\partial r} = 0$$

$$\frac{\partial f}{\partial r} = 1 \qquad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - mg\cos\theta \qquad \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$0 = mr\dot{\theta}^2 - mg\cos\theta - m\ddot{r} + \lambda = 0$$

<u>θ EOM</u>

$$\frac{\partial f}{\partial \theta} = 0 \qquad \frac{\partial L}{\partial \theta} = mgr\sin\theta \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

$$0 = mgr\sin\theta - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} + \lambda \cdot 0 = 0$$

Impose constraint $r = a$, $\dot{r} = 0$, $\ddot{r} = 0$

<u>θ EOM</u>

$$mga\sin\theta - ma^2\ddot{\theta} = 0$$

$$\ddot{\theta} = \frac{g}{a}\sin\theta$$

r EOM

$$ma\dot{\theta}^2 - mg\cos\theta + \lambda = 0$$

$$\lambda = mg\cos\theta - ma\dot{\theta}^2$$

Integrate θ EOM

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$

$$\dot{\theta}\frac{d\dot{\theta}}{d\theta} = \frac{g}{a}\sin\theta$$

$$\int_0^{\dot{\theta}} \dot{\theta}\, d\dot{\theta} = \frac{g}{a}\int_0^{\theta} \sin\theta\, d\theta$$

$$\frac{1}{2}\dot{\theta}^2 = -\frac{g}{a}\cos\theta\Big|_0^{\theta} = \frac{g}{a}\left(1 - \cos\theta\right)$$

$$\dot{\theta}^2 = \frac{2g}{a}\left(1 - \cos\theta\right)$$

Substitute Into λ equation

$$\lambda = mg\cos\theta - ma\dot{\theta}^2$$

$$= mg\cos\theta - ma\frac{2g}{a}(1-\cos\theta)$$

$$= mg\cos\theta(1+2) - 2mg$$

$$= mg(3\cos\theta - 2)$$

The force of constraint is

$$Q_r = \lambda\frac{\partial f}{\partial r} = mg(3\cos\theta - 2)$$

The bead leaves the surface when $Q_r = 0$

$$\cos\theta = 2/3 \qquad \theta = 48^0$$

10.26 Free particle under gravity

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \qquad V = mgz$$

$$L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

<u>Canonical Momenta</u>

$$P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$P_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$$

$$P_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

<u>Hamiltonian</u> $\quad H = T + V$

$$= \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{P_z^2}{2m} + mgz$$

Hamilton's Equations

$$\frac{\partial H}{\partial p_x} = \dot{x} = \frac{p_x}{m}$$

$$\frac{\partial H}{\partial p_y} = \dot{y} = \frac{p_y}{m}$$

$$\frac{\partial H}{\partial p_z} = \dot{z} = \frac{p_z}{m}$$

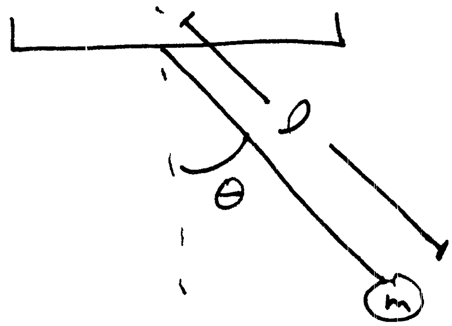
$$\frac{\partial H}{\partial x} = -\dot{p}_x = 0$$

$$\frac{\partial H}{\partial y} = -\dot{p}_y = 0$$

$$\frac{\partial H}{\partial z} = -\dot{p}_z = mg$$

All check.

10.27



$$T = \frac{1}{2} m (l\dot{\theta})^2 \qquad V = mgl(1-\cos\theta)$$

$$L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta)$$

<u>Momenta</u>

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \qquad \dot{\theta} = \frac{p_\theta}{m l^2}$$

<u>Hamiltonian</u>

$$H = T + V = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1-\cos\theta)$$

$$= \frac{1}{2} m l^2 \left(\frac{p_\theta}{m l^2}\right)^2 + mgl(1-\cos\theta)$$

$$= \frac{1}{2} \frac{p_\theta^2}{m l^2} + mgl(1-\cos\theta)$$

EOM

$$\frac{\partial H}{\partial p_\theta} = \dot{\theta} = \frac{p_\theta}{m l^2} \qquad p_\theta = m l^2 \dot{\theta}$$

$$\frac{\partial H}{\partial \theta} = -\dot{p}_\theta = m g l \sin\theta$$

Using both

$$-m l^2 \ddot{\theta} = m g l \sin\theta$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$