Homework 7

Due Tuesday 11/10/2009 at 5:30pm in my box in physics. These may also be handed in at the end of Justin Mitchell's office hours in PHYS 228 from 4:00-5:30pm Tuesday or at the SPS meeting starting at 5:30pm in PHYS 134.

Fowles Problems

10.13

10.18

11.1

11.2

11.18 Find only normal frequencies.

11.20

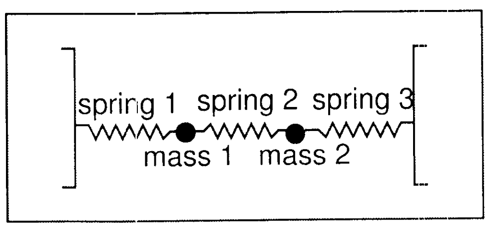
E1 The system below has two masses each of mass m and three springs with spring constants $k_1 = k$, $k_2 = 2k$, and $k_3 = 3k$. Let the equilibrium location of the masses be given by x_1 and x_2, measured from the left support. The masses oscillate in a line.

(a) Write the Lagrangian for the system.

(b) Find the frequencies of the normal modes of the system.

(c) Find the eigenvectors of the normal modes.

(d) How must the system be prepared for it to oscillate only with the lowest normal frequency?



where $0 \le \theta \le \pi$ and a is a constant. Find the Lagrangian function and the equation of motion of the particle.

10.12 A simple pendulum of length l and mass m is suspended from a point on the circumference of a thin massless disc of radius a that rotates with a constant angular velocity ω about its central axis as shown in Figure P10.12. Find the equation of motion of the mass m.

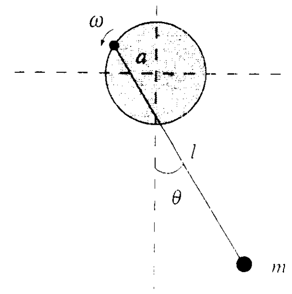


Figure P10.12

10.13 A bead of mass m is constrained to slide along a thin, circular hoop of radius l that rotates with constant angular velocity ω in a horizontal plane about a point on its rim as shown in Figure P10.13.

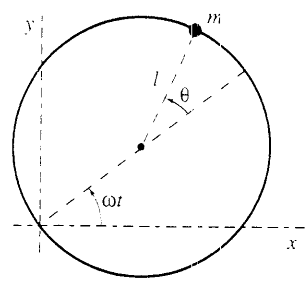


Figure P10.13

(**a**) Find Lagrange's equation of motion for the bead.
(**b**) Show that the bead oscillates like a pendulum about the point on the rim diametrically opposite the point about which the hoop rotates.
(**c**) What is the effective "length" of this "pendulum"?

10.14 The point of support of a simple pendulum is being elevated at a constant acceleration a, so that the height of the support is $\frac{1}{2}at^2$, and its vertical velocity is at. Find the differential equation of motion for small oscillations of the pendulum by Lagrange's method. Show that the period of the pendulum is $2\pi[l/(g+a)]^{1/2}$, where l is the length of the pendulum.

10.15 Work Problem 8.12 by using the method of Lagrange multipliers. (**a**) Show that the acceleration of the ball is $\frac{5}{7}g$. (**b**) Find the tension in the string.

10.16 A heavy elastic spring of uniform stiffness and density supports a block of mass m. If m' is the mass of the spring and k its stiffness, show that the period of vertical oscillations is

$$2\pi\sqrt{\frac{m+(m'/3)}{k}}$$

This problem shows the effect of the mass of the spring on the period of oscillation. (*Hint: To set up the Lagrangian function for the system, assume that the velocity of any part of the spring is proportional to its distance from the point of suspension.*)

10.17 Use the mathed of Lagrange multipliers to find the tensions in the two strings of the double Atwood machine of Example 10.5.4.

10.18 A smooth rod of length l rotates in a plane with a constant angular velocity ω about an axis fixed at one end of the rod and perpendicular to the plane of rotation. A bead of mass m is initially positioned at the stationary end of the rod and given a slight push such that its initial speed directed along the rod is ωl.

(a) Find the time it takes the bead to reach the other end of the rod.

(b) Use the method of Lagrange multipliers to find the reaction force **F** that the rod exerts on the bead.

10.19 A particle of mass m perched on top of a smooth hemisphere of radius a is disturbed ever so slightly, so that it begins to slide down the side. Find the normal force of constraint exerted by the hemisphere on the particle and the angle relative to the vertical at which it leaves the hemisphere. Use the method of Lagrange multipliers.

10.20 A particle of mass m_1 slides down the smooth circular surface of radius of curvature a of a wedge of mass m_2 that is free to move horizontally along the smooth horizontal surface on which it rests (Figure P10.20).

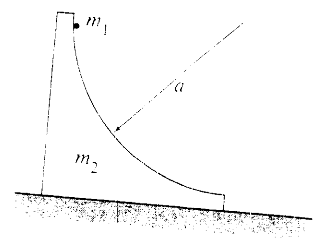


Figure P10.20

(a) Find the equations of motion for each mass.

(b) Find the normal force of constraint exerted by the wedge on the particle. Use the method of Lagrange multipliers.

10.21 **(a)** Find the general differential equations of motion for a particle in cylindrical coordinates R, ϕ, z. Use the relation

$$\begin{aligned} v^2 &= v_R^2 + v_\phi^2 + v_z^2 \\ &= \dot{R}^2 + R^2\dot{\phi}^2 + \dot{z}^2 \end{aligned}$$

(b) Find the general differential equations of motion for a particle in spherical coordinates: r, θ, ϕ. Use the relation

$$\begin{aligned} v^2 &= v_r^2 + v_\theta^2 + v_\phi^2 \\ &= \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta \end{aligned}$$

(**Note:** Compare your results with the result derived in Chapter 1, Equations 1.12.3 and 1.12.14 by setting $\mathbf{F} = m\mathbf{a}$ and taking components.)

10.22 Find the differential equations of motion in *three* dimensions for a particle in a central field using spherical coordinates.

10.23 A bar of soap slides in a smooth bowl in the shape of an inverted right circular cone of apex angle 2α. The axis of the cone is vertical. Treating the bar of soap as a particle of mass m find the differential equations of motion using spherical coordinates with $\theta = \alpha$ = constant. As is the case with the spherical pendulum, Example 10.6.2, show that the particle, given an initial motion with $\dot{\phi}_0 \neq 0$, must remain between two horizontal circles on the cone.

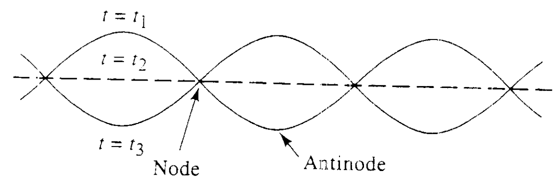


Figure 11.6.3 A standing sinusoidal wave.

are illustrated in Figure 11.6.3. Note again that there is a well-defined constraint on the values of allowable wavelengths λ. Because the endpoints of the string are fixed, we have as boundary conditions

$$q = 0 \qquad (x = 0, L) \tag{11.6.15}$$

that our solution (Equation 11.6.14) must obey. The first condition at $x = 0$ is met automatically. The second boundary condition at $x = L$ is met if

$$L = N\left(\frac{\lambda}{2}\right) \qquad \lambda = \frac{2L}{N} \tag{11.6.16}$$

An integral number of half wavelengths must fit within the length L if the endpoints are to be nodes. This is precisely the condition obtained previously for the normal modes of the loaded string.

Problems

11.1 A particle of mass m moves in one-dimensional motion with the following potential energy functions:

(a) $V(x) = \frac{k}{2}x^2 + \frac{k^2}{x}$

(b) $V(x) = kxe^{-bx}$

(c) $V(x) = k(x^4 - b^2x^2)$

where all constants are real and positive. Find the equilibrium positions for each case and determine their stability.

(d) Find the angular frequency ω for small oscillations about the respective positions of stable equilibrium for parts (a), (b), and (c), and find the period in seconds for each case if $m = 1$ g, and k and b are each of unit value in cgs units.

11.2 A particle moves in two dimensions under the potential energy function

$$V(x,y) = k(x^2 + y^2 - 2bx - 4by)$$

where k is a positive constant. Show that there is one position of equilibrium. Is it stable or unstable?

11.3 The potential energy function of a particle of mass m in one-dimensional motion is given by

$$V(x) = -\frac{k}{2}x^2$$

as follows:

$$x_1(t) = \frac{1}{2}A_0(\cos\omega_a t + \cos\omega_b t) = A_0\cos\overline{\omega}t\ \cos\Delta t$$

$$x_2(t) = \frac{1}{2}A_0(\cos\omega_a t - \cos\omega_b t) = A_0\ \sin\overline{\omega}t\ \sin\Delta t$$

in which $\overline{\omega} = (\omega_a + \omega_b)/2$ and $\Delta = (\omega_b - \omega_a)/2$. Thus, if the coupling is very weak so that $K' \ll K$, then $\overline{\omega}$ will be very nearly equal to $\omega_a = (K/m)^{1/2}$, and Δ is very small. Consequently, under the stated initial conditions, the first oscillator eventually comes to rest while the second oscillator oscillates with amplitude A_0. Later, the system returns to the initial condition, and so on. Thus, the energy passes back and forth between the two oscillators indefinitely.

11.14 In Problem 11.13 show that, for weak coupling, the period at which the energy trades back and forth is approximately equal to $T_a(2K/K')$ where $T_a = 2\pi/\omega_a = 2\pi/(m/K)^{1/2}$ is the period of the symmetric oscillation.

11.15 Two identical simple pendula are coupled together by a very weak force of attraction that varies as the inverse square of the distance between the two particles. (This force might be the gravitational attraction between the two particles, for instance.) Show that, for small departures from the equilibrium configuration, the Lagrangian can be reduced to the same mathematical form, with appropriate constants, as that of the two identical coupled oscillators treated in Section 11.3 and in Problem 11.13. (*Hint: Consider Equation 11.3.9.*)

11.16 Find the normal frequencies of the coupled harmonic oscillator system (see Figure 11.3.1) for the general case in which the two particles have unequal mass and the springs have different stiffness. In particular, find the frequencies for the case $m_1 = m$, $m_2 = 2m$, $K_1 = K$, $K_2 = 2K$, $K' = 2K$. Express the result in terms of the quantity $\omega_0 = (K/m)^{1/2}$.

11.17 A light elastic spring of stiffness K is clamped at its upper end and supports a particle of mass m at its lower end. A second spring of stiffness K is fastened to the particle and, in turn, supports a particle of mass $2m$ at its lower end. Find the normal frequencies of the system for vertical oscillations about the equilibrium configuration. Find also the normal coordinates.

11.18 Consider the case of a double pendulum, Figure 11.3.7a, in which the two sections are of different length, the upper one being of length l_1 and the lower of length l_2. Both particles are of equal mass m. Find the normal frequencies of the system and the normal coordinates.

11.19 Set up the secular equation for the case of three coupled particles in a linear array and show that the normal frequencies are the same as those given by Equation 11.5.17.

11.20 A simple pendulum of mass m and length a is attached to a block of mass M that is constrained to slide along a frictionless, horizontal track as shown in Figure P11.20. Find the normal frequencies and normal modes of oscillation.

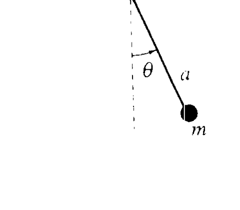
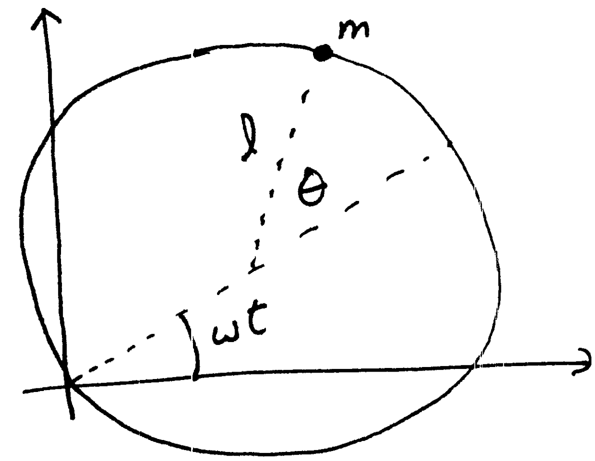


Figure P11.20

10.13



Radius $= l$

Let (x_c, y_c) be the center of the circle, and (x,y) be the location of the masses.

$$x_c = l\cos\omega t \qquad y_c = l\sin\omega t$$

$$x = x_c + l\cos(\theta+\omega t) \qquad y = y_c + l\cos(\theta+\omega t)$$

$$\dot{x} = -l\omega\sin\omega t - l\sin(\theta+\omega t)(\dot\theta+\omega)$$

$$\dot{y} = l\omega\cos\omega t + l\cos(\theta+\omega t)(\dot\theta+\omega)$$

$$T = \tfrac{1}{2}m(\dot{x}^2+\dot{y}^2) = \tfrac{1}{2}m\Big(l^2\omega^2 + l^2(\dot\theta+\omega)^2$$
$$+ 2l^2\omega\sin\omega t\sin(\theta+\omega t)(\dot\theta+\omega)$$
$$+ 2l^2\omega\cos\omega t\cos(\theta+\omega t)(\dot\theta+\omega)\Big)$$

$$\cos(\theta + \omega t - \omega t) = \cos(\theta+\omega t)\cos\omega t + \sin(\theta+\omega t)\sin\omega t$$

$$= \cos\theta$$

From A-6.

$$T = \frac{1}{2} m l^2 \left[\omega^2 + (\dot\theta + \omega)^2 + 2\omega(\dot\theta+\omega)\cos\theta\right]$$

$$= L \qquad \text{since } V = 0$$

$$\frac{\partial L}{\partial \theta} = \frac{1}{2} m l^2 \left(-2\omega(\dot\theta+\omega)\sin\theta\right)$$

$$\frac{\partial L}{\partial \dot\theta} = \frac{1}{2} m l^2 \left(2(\dot\theta+\omega) + 2\omega\cos\theta\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = \frac{1}{2} m l^2 \left(2\ddot\theta - 2\omega\sin\theta\,\dot\theta\right)$$

EOM

$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot\theta} = m l^2 \left[-\omega(\dot\theta+\omega)\sin\theta - \ddot\theta + \omega\sin\theta\,\dot\theta\right]$$

$$\ddot{\Theta} + \omega^2 \sin\Theta = 0$$

(b) For small θ, frequency is ω^2.

$$\ddot{\Theta} + \omega^2 \Theta = 0$$

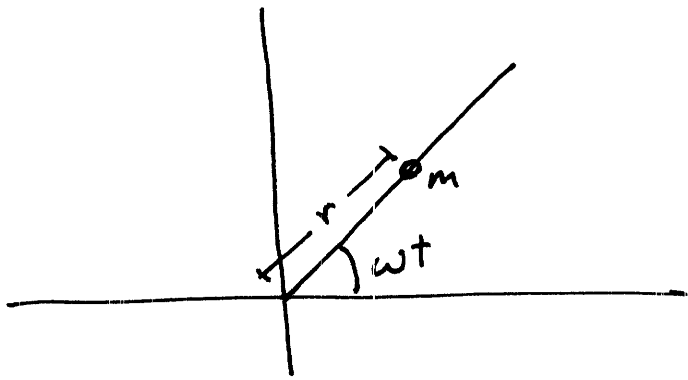
$\Theta = 0$ is the point opposite the pivot.

(c) If the ring was a pendulum

$$\omega^2 = g/l$$

$$l = g/\omega^2$$

(10.18) $V_0 = \omega l$



$$T = \frac{1}{2} m \vec{v} \cdot \vec{v} = \frac{1}{2} m \left(\dot{r}^2 + (r\dot{\theta})^2 \right)$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \omega^2 \right) \quad = L$$

$V = 0$

<u>r EOM</u>

$$\frac{\partial L}{\partial r} = m r \omega^2 \qquad \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = m \ddot{r}$$

$$\frac{\partial L}{\partial r} - \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = m r \omega^2 - m \ddot{r} = 0$$

$$\ddot{r} - \omega^2 r = 0$$

Now find force of constraint $Q_\theta = r F_\theta$

Constraint $\quad f = \theta - \omega t = 0 \qquad \frac{\partial f}{\partial r} = 0 \qquad \frac{\partial f}{\partial \theta} = 1$

<u>EOM r</u> $\qquad T = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 = L$

$$\frac{\partial L}{\partial r} = m r \dot{\theta}^2 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m \ddot{r}$$

$$0 = \frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} + \lambda(t)\frac{\partial f}{\partial r} = 0$$

$$m r \dot{\theta}^2 - m \ddot{r} = 0$$

<u>EOM θ</u>

$$\frac{\partial L}{\partial \theta} = 0 \qquad \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -m r^2 \ddot{\theta} - 2 m r \dot{r} \dot{\theta} = 0$$

<u>Constraint</u> $\qquad \theta = \omega t$

$$\dot{\theta} = \omega$$

$$\ddot{\theta} = 0$$

$$\frac{d\dot{r}}{dt} = \frac{d\dot{r}}{dr}\frac{dr}{dt} = \dot{r}\frac{d\dot{r}}{dr} = \omega^2 r$$

$$\int_{v_0}^{v} \dot{r}\, d\dot{r} = \int_0^r \omega^2 r\, dr$$

$$\frac{1}{2}\left(\dot{r}^2 - v_0^2\right) = \frac{1}{2}\omega^2 r^2$$

$$\dot{r} = \sqrt{v_0^2 + \omega^2 r^2} = \frac{dr}{dt}$$

$$v_0 = \omega \ell$$

Integrate from $0 \to T$

$$\frac{dr}{dt} = \omega\sqrt{\ell^2 + r^2}$$

$$\int_0^{\ell} \frac{dr}{\sqrt{\ell^2 + r^2}} = \int_0^T \omega\, dt = \omega T$$

$$\ln\left(1 + \sqrt{2}\right) = \omega T$$

$$T = \frac{1}{\omega}\ln\left(1 + \sqrt{2}\right)$$

EOM θ

$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda \frac{\partial f}{\partial \theta} = 0$$

$$= 0 - mr^2\ddot{\theta} - 2mr\dot{r}\dot{\theta} + \lambda = 0$$

Apply constraint $\theta = \omega t \quad \dot{\theta} = \omega \quad \ddot{\theta} = 0$

$$\lambda = 2mr\dot{r}\omega$$

$$\dot{r} = \omega\sqrt{l^2 + r^2}$$ from earlier.

$$\lambda = 2mr\omega^2\sqrt{l^2+r^2}$$

This is the torque exerted by the rod (unit Nm).

The normal force is

$$N = \frac{\lambda}{r} = 2m\omega^2\sqrt{l^2+r^2}$$