11.1 (a)

$$V = \frac{kx^2}{2} + \frac{k^2}{x}$$

$$\frac{\partial V}{\partial x} = kx - \frac{k^2}{x^2} = 0$$

$$x_0 = k^{1/3}$$

$$V_0'' = \left.\frac{\partial^2 V}{\partial x^2}\right|_{x_0} = \left. k + \frac{2k^2}{x^3}\right|_{x_0} = k + 2k = 3k > 0$$

$$\omega_0^2 = \frac{V_0''}{m} = \frac{3k}{m} = \left(\frac{2\pi}{T}\right)^2$$

$$T = \cancel{\frac{2\pi m}{3k}} = 2\pi\sqrt{\frac{m}{3k}} = \frac{2\pi}{\sqrt{3}} \text{ sec}$$

(b)

(b) $V = kxe^{-bx}$

$$\frac{\partial V}{\partial x} = ke^{-bx} - kbxe^{-bx} = 0$$

$$x_0 = \frac{1}{b} \quad \text{for equilibrium}$$

$$V_0'' = \left.\frac{\partial^2 V}{\partial x^2}\right|_{x_0} = \left. -bke^{-bx} + kb^2xe^{-bx} - kbe^{-bx} \right|_{x_0}$$

$$= \frac{-2kb}{e} + \frac{kb}{e} = -\frac{kb}{e} < 0 \quad \text{Unstable}$$

No ω_0 for unstable equilibrium.

(c) $V = k(x^4 - b^2x^2)$

$$\frac{\partial V}{\partial x} = 0 = k(4x^3 - 2b^2x) = 0$$

$$x_0 = \pm\frac{b}{\sqrt{2}} \quad \text{Equilibrium}$$

$$= 0$$

$$\pm\frac{b}{\sqrt{2}} \qquad \left.\frac{\partial^2 V}{\partial x^2}\right|_{x_0} = k(12x^2 - 2b^2) = k(6b^2 - 2b^2)$$

$$= 4kb^2 > 0 \quad \text{stable}$$

$$V_0'' = 4kb^2$$

$$\omega_0^2 = \frac{4kb^2}{m} = 4$$

$$\omega_0 = 2 = \frac{2\pi}{T}$$

$T = \pi$ seconds

$\underline{x_0 = 0}$ $$\frac{\partial^2 V}{\partial x^2} = -2kb^2 < 0 \quad \text{Unstable}$$

(11.2) $$V(x,y) = k(x^2+y^2-2bx-4by)$$

$$q_1 = x,\ q_2 = y$$

Equilibrium

$$\frac{\partial V}{\partial x} = 0 = k(2x-2b) \qquad x_0 = b$$

$$\frac{\partial V}{\partial y} = 0 = k(2y-4b) \qquad y_0 = 2b$$

$$K_{11} = \left.\frac{\partial^2 V}{\partial x^2}\right|_{q_0} = 2k \qquad K_{12} = K_{21} = \frac{\partial^2 V}{\partial x \partial y} = 0$$

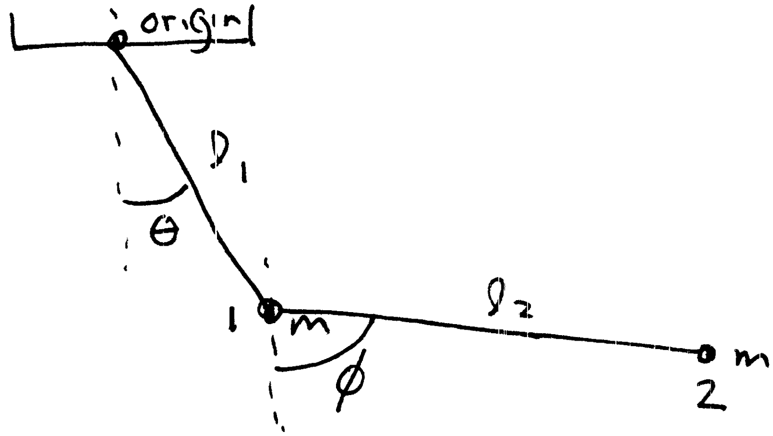
$$K_{22} = \left.\frac{\partial^2 V}{\partial y^2}\right|_{q_0} = 2k$$

Stability

$$K_{11} > 0 \qquad \begin{vmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{vmatrix} = K_{11}K_{22} = 4k^2 > 0$$

stable

11.18



$$x_1 = l_1 \sin\theta$$

$$y_1 = -l_1 \cos\theta$$

$$x_2 = x_1 + l_2 \sin\phi = l_1 \sin\theta + l_2 \sin\phi$$

$$y_2 = y_1 - l_2 \cos\phi = -l_1 \cos\theta - l_2 \cos\phi$$

$$\dot{x}_1 = l_1 \cos\theta\,\dot{\theta} \qquad \dot{y}_1 = l_1 \sin\theta\,\dot{\theta}$$

$$\dot{x}_2 = l_1 \cos\theta\,\dot{\theta} + l_2 \cos\phi\,\dot{\phi}$$

$$\dot{y}_2 = l_1 \sin\theta\,\dot{\theta} + l_2 \sin\phi\,\dot{\phi}$$

$$T_1 = \frac{1}{2} m\left(\dot{x}_1^2 + \dot{y}_1^2\right) = \frac{1}{2} m l_1^2 \dot{\theta}^2$$

$$T_2 = \frac{1}{2} m \left(\dot{x}_2^2 + \dot{y}_2^2 \right)$$

$$= \frac{1}{2} m \Big(l_1^2 \dot{\theta}^2 + l_2 \dot{\phi}^2$$

$$+ 2 l_1 l_2 \dot{\theta} \dot{\phi} \underbrace{\left(\cos\theta \cos\phi + \sin\theta \sin\phi \right)}_{\cos(\theta - \phi)} \Big)$$

$$= \frac{1}{2} m \left(l_1^2 \dot{\theta}^2 + l_2^2 \dot{\phi}^2 + 2 l_1 l_2 \dot{\theta} \dot{\phi} \cos(\theta - \phi) \right)$$

$$T = T_1 + T_2$$

$$= \frac{1}{2} m \left(2 l_1^2 \dot{\theta}^2 + l_2^2 \dot{\phi}^2 + 2 l_1 l_2 \dot{\theta} \dot{\phi} \cos(\theta - \phi) \right)$$

$$V = m g y_1 + m g y_2$$

$$= - m g l_1 \cos\theta - m g l_1 \cos\theta - m g l_2 \cos\phi$$

Expand to $O(\theta^2)$

$$\cos(\cancel{x}) = 1 - \frac{1}{2}x^2$$

$$T = \frac{1}{2}m\left(2l_1^2\dot{\theta}^2 + l_2^2\dot{\phi}^2 + 2l_1 l_2\dot{\theta}\dot{\phi}\right)$$

$$V = -2mgl_1\left(1-\frac{1}{2}\theta^2\right) - mgl_2\left(1-\frac{1}{2}\phi^2\right)$$

$$= \cancel{+}mgl_1\theta^2 + \frac{1}{2}mgl_2\phi^2$$

Discarding constant terms in the potential.

$$L = T - V = \frac{1}{2}m\Big(\underbrace{2l_1^2}_{M_{11}}\dot{\theta}^2 + \underbrace{l_2^2}_{M_{22}}\dot{\phi}^2 + \underbrace{2l_1l_2}_{2M_{12}}\dot{\theta}\dot{\phi}\Big)$$

$$- \underbrace{mg(l_1}_{K_{11}}\theta^2 + \underbrace{\tfrac{1}{2}l_2}_{K_{22}}\phi^2)$$

$$L = \frac{1}{2}\sum M_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum K_{ij}q_iq_j$$

$M_{11} = 2ml_1^2$ $\qquad M_{22} = ml_2^2$ $\qquad M_{12} = l_1 l_2 m$

$K_{11} = 2mgl_1$ $\qquad K_{22} = mgl_2$ $\qquad K_{12} = 0$

Read from Lagrangian

The matrix representing the EOM

$$\begin{pmatrix} K_{11} - \omega^2 M_{11} & K_{12} - \omega^2 M_{12} \\ K_{21} - \omega^2 M_{21} & K_{22} - \omega^2 M_{22} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2mgl_1 - \omega^2 2ml_1^2 & -\omega^2 l_1 l_2 m \\ -\omega^2 l_1 l_2 m & mgl_2 - \omega^2 m l_2^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

Since $= 0$, can factor common terms out of each row.

$$\begin{pmatrix} 2g - 2\omega^2 l_1 & -\omega^2 l_2 \\ -\omega^2 l_1 & g - \omega^2 l_2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

Solvable if $\det() = 0$

$$(2g - 2\omega^2 l_1)(g - \omega^2 l_2) - \omega^4 l_1 l_2 = 0$$

$$2g^2 - 2\omega^2 g l_1 - 2\omega^2 g l_2 + \omega^4 l_1 l_2 = 0$$

$$2g^2 - 2\omega^2 g (l_1 + l_2) + \omega^4 l_1 l_2 = 0$$

<u>Quadratic Formula</u>

$$\omega^2 = \frac{2g(l_1+l_2) \pm \sqrt{4g^2(l_1+l_2)^2 - 8g^2 l_1 l_2}}{2 l_1 l_2}$$

$$\sqrt{} = 2g\sqrt{l_1^2 + 2l_1 l_2 + l_2^2 - 2l_1 l_2}$$

$$= 2g\sqrt{l_1^2 + l_2^2}$$

The normal mode frequencies are

$$\omega^2 = \frac{g}{l_1 l_2}\left(l_1 + l_2 \pm \sqrt{l_1^2 + l_2^2} \right)$$

Two degrees of freedom $\Rightarrow$ Two modes.

Alternate Solution   Go back to L and write EOM

EOM θ

$$0 = \frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -2mgl_1\theta - 2ml_1^2\ddot{\theta} - ml_1l_2\ddot{\phi}$$

EOM ϕ

$$0 = \frac{\partial L}{\partial \phi} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} = -mgl_2\phi - ml_2^2\ddot{\phi} - ml_1l_2\ddot{\theta}$$

Substitute

$$\theta(t) = A_1\cos(\omega t+\sigma) \qquad \phi(t) = A_2\cos(\omega t+\sigma)$$

and cancel $\cos(\omega t+\sigma)$

EOM θ

$$-2mgl_1A_1 + 2ml_1^2\omega^2A_1 + ml_1l_2\omega^2A_2 = 0$$

EOM ϕ

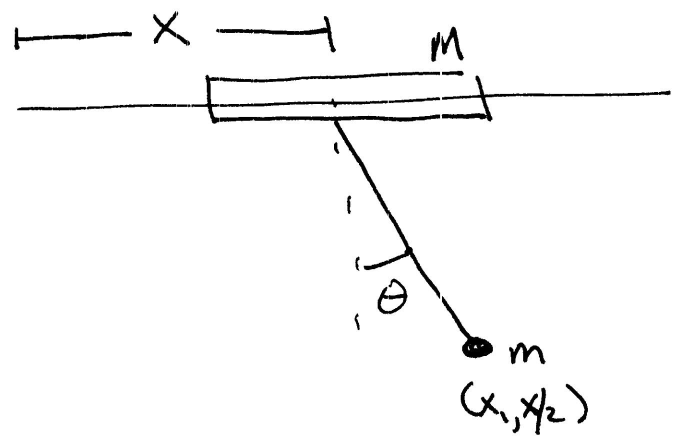
$$-mgl_2A_2 + m\omega^2l_2^2A_2 + m\omega^2l_1l_2A_1 = 0$$

Write as matrix

$$\begin{pmatrix} -2mgl_1 + 2ml_1^2\omega^2 & ml_1l_2\omega^2 \\ ml_1l_2\omega^2 & -mgl_2 + m\omega^2 l_2^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

Correct up to a global sign that doesn't matter.

(11.20)



We are expecting one $\omega = 0$ modes, since the system can translate without oscillating.

$$T = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m (\dot{x}_1^2 + \dot{y}_2^2)$$

$$x_1 = X + l \sin\theta \qquad y_2 = -l \cos\theta$$

$$\dot{x}_1 = \dot{X} + l \cos\theta \dot{\theta} \qquad \dot{y}_2 = l \sin\theta \dot{\theta}$$

$$T = \frac{1}{2} M \dot{X}^2 + \frac{1}{2} m \left(\dot{X}^2 + 2 l \dot{X} \dot{\theta} \cos\theta + l^2 \dot{\theta}^2 \right)$$

$$V = m g y_2 = -m g l \cos\theta$$

$$L = T - V = \frac{1}{2}(M+m)\dot{X}^2 + m l \dot{X} \dot{\theta} \cos\theta + \frac{1}{2} m l^2 \dot{\theta}^2$$
$$+ m g l \cos(\theta)$$

Expand L to $O(x^2)$ $\qquad \cos x \approx 1 - \frac{1}{2}x^2$

$$L = \frac{1}{2}(M+m)\dot{x}^2 + ml\dot{x}\dot{\theta} + \frac{1}{2}ml^2\dot{\theta}^2$$
$$+ mgl\left(1 - \frac{1}{2}\theta^2\right)$$

EOM x (Ignorable Coordinate)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 = \frac{d}{dt}\Big(\underbrace{(M+m)\dot{x} + ml\dot{\theta}}_{\text{momentum of center of mass}}\Big)$$

$$= (M+m)\ddot{x} + ml\ddot{\theta} = 0$$

EOM θ

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = -mgl\theta - ml\ddot{x} - ml^2\ddot{\theta} = 0$$

$$\ddot{x} + l\ddot{\theta} + g\theta = 0$$

Propose $\quad x(t) = A_1 \cos(\omega t + \sigma)$

$$\theta(t) = A_2 \cos(\omega t + \sigma)$$

EOM x

$$-\omega^2 A_1 (M+m) - \omega^2 m l A_2 = 0$$

EOM θ

$$-\omega^2 A_1 - l\omega^2 A_2 + g A_2 = 0$$

$$\begin{pmatrix} -\omega^2(M+m) & -\omega^2 m l \\ -\omega^2 & -l\omega^2 + g \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$\det() = 0 = -\omega^2(M+m)(g - l\omega^2) - \omega^4 m l = 0$$

$\omega = 0$ is solution

$$(M+m)g - l(M+m)\omega^2 + m l \omega^2 = 0$$

$$(M+m)g - l M \omega^2 = 0$$

Normal Mode Frequencies

$$\omega_1^2 = 0 \qquad \omega_2^2 = \frac{g}{l}\left(\frac{M+m}{M}\right)$$

Find Normal Coordinates - ~~Frequencies~~ Eigenvectors

$\underline{\omega_1^2 = 0}$ - Sub back into matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$A_2 = 0$$

$$x(t) = A_1 \cos(\delta) = A_1'$$

$$\theta(t) = 0$$

But this does not solve EOM ($\omega^2 = 0$ is a special case)

$$(M+m)\ddot{x} + ml\ddot{\theta} = 0 = (M+m)\ddot{x} \quad \text{if } \theta = 0$$

But we can solve $\ddot{x} = 0 \Rightarrow x(t) = A + Bt$

$$Q_1 = \begin{pmatrix} A + Bt \\ 0 \end{pmatrix}$$

$$\omega_2^2 = \frac{g}{l}\left(\frac{M+m}{M}\right)$$

$$\begin{pmatrix} -\omega_2^2(M+m) & -\omega_2^2 m l \\ -\omega_2^2 & -l\omega_2^2 + g \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

Look at second row

$$-\omega_2^2 A_1 \quad - l\omega_2^2 A_2 \ + g A_2 = 0$$

Assume $A_1 = 1$

$$A_2 = \frac{-\omega_2^2}{l\omega_2^2 - g} = - \frac{\frac{g}{l}\left(\frac{M+m}{M}\right)}{l\frac{g}{l}\left(\frac{M+m}{M}\right) - g}$$

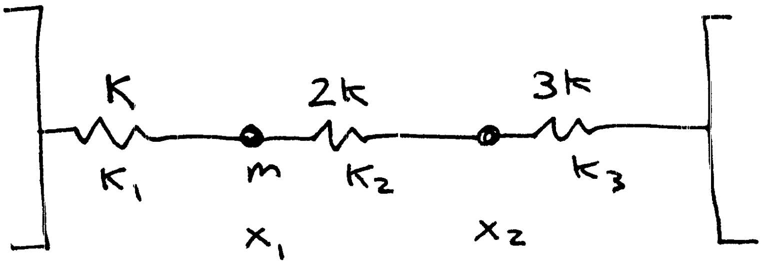
$$= -\frac{1}{l} \frac{M+m}{M+m-M}$$

$$= -\frac{1}{l} \frac{M+m}{M}$$

$$Q_2 = \begin{pmatrix} 1 \\ -\dfrac{M+m}{\ell M} \end{pmatrix}$$

Note the wierd dimensional problems coming from one linear and one angular coordinate.

E1



$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_3 x_2^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$$

$$= \frac{1}{2} \left(k_1 x_1^2 + k_2 x_1^2 + k_2 x_2^2 + k_3 x_2^2 - 2 k_2 x_1 x_2 \right.$$

$$= \frac{k}{2} \left(3 x_1^2 + 5 x_2^2 - 4 x_1 x_2 \right)$$

$$L = T - V = \frac{1}{2} m \left(\dot{x}_1^2 + \dot{x}_2^2 \right) - \frac{k}{2} \left(3 x_1^2 + 5 x_2^2 - 4 x_1 x_2 \right)$$

x_1 EOM

$$-3kx_1 + 2kx_2 - m\ddot{x}_1 = 0$$

x_2 EOM

$$-5kx_2 + 2kx_1 - m\ddot{x}_2 = 0$$

Propose Solution

$$x_1(t) = A_1 \cos(\omega t + \delta) \qquad x_2(t) = A_2 \cos(\omega t + \delta)$$

x_1 EOM

$$-3kA_1 + 2kA_2 + m\omega^2 A_1 = 0$$

x_2 EOM

$$-5kA_2 + 2kA_1 + m\omega^2 A_2 = 0$$

Matrix

$$\begin{pmatrix} m\omega^2 - 3k & 2k \\ 2k & m\omega^2 - 5k \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

Define $\omega_0^2 = k/m$

$$\begin{pmatrix} \omega^2 - 3\omega_0^2 & 2\omega_0^2 \\ 2\omega_0^2 & \omega^2 - 5\omega_0^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$\det(\,) = 0$

$$(\omega^2 - 3\omega_0^2)(\omega^2 - 5\omega_0^2) - 4\omega_0^4 = 0$$

$$\omega^4 - 8\omega_0^2\omega^2 + 11\omega_0^4 = 0$$

<u>Quadratic Formula</u>

$$\omega^2 = \frac{8\omega_0^2 \pm \sqrt{64\omega_0^4 - 44\omega_0^4}}{2}$$

$$= (4 \pm \sqrt{5})\,\omega_0^2$$

Normal Coordinates $\omega_1^2 = (4+\sqrt{5})\omega_0^2$

Sub into matrix

$$\begin{pmatrix} (4+\sqrt{5}-3)\omega_0^2 & 2\omega_0^2 \\ 2\omega_0^2 & (4+\sqrt{5}-5)\omega_0^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$(1+\sqrt{5})A_1 + 2A_2 = 0$$

If $A_1 = 1$, $A_2 = -\frac{1}{2}(1+\sqrt{5})$

$$Q_1 = \begin{pmatrix} 1 \\ -\frac{1}{2}(1+\sqrt{5}) \end{pmatrix}$$

Normal Coordinates $\omega_2^2 = (4-\sqrt{5})\omega_0^2$

$$\begin{pmatrix} (4-\sqrt{5}-3)\omega_0^2 & 2\omega_0^2 \\ 2\omega_0^2 & (4-\sqrt{5}-5)\omega_0^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

$$(1-\sqrt{5})A_1 + 2A_2 = 0$$

If $A_1 = -\frac{(1-\sqrt{5})}{2}$

$$Q_2 = \begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix}$$

To oscillate ~~about~~ with lowest frequency $\omega_2 = 4-\sqrt{5}$ prepare system with Q_2 displacements, $x_1(\cancel{t}0) = 1$

$$x_2(0) = \frac{1}{2}(\sqrt{5}-1)$$