Homework 8

Due Tuesday 11/17/2009 at 5:30pm in my box in physics. These may also be handed in at the end of Justin Mitchell's office hours in PHYS 228 from 4:00-5:30pm Tuesday.

Fowles Problems

6.1

6.3 Note, Justin worked this out on the practice test. Now you need to work it, a very cool problem.

6.5

6.9

6.10

6.16

6.18

E1 Compute the gravitational field strength 1cm from the center of a thin linear object a distance 1m long with linear mass density $\lambda = 5\text{kg/m}$. Check by assuming the object is infinitely long and applying Gauss' law for gravity. Compute the gravitational potential as a function of distance along a line perpendicular to the object. Show your field strength is the appropriate derivative.

6.1

$$\rho = 11.35 \frac{\text{gm}}{\text{cm}^3} \cdot \frac{1\,\text{kg}}{1000\,\text{gm}} \cdot \left(\frac{100\,\text{cm}}{\text{m}}\right)^3$$

$$= 11,350 \frac{\text{kg}}{\text{m}^3}$$

Radius of sphere

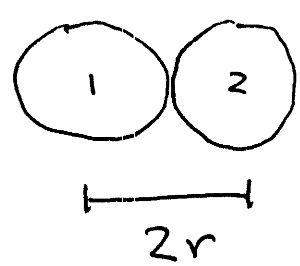
$$m = \frac{4}{3}\pi \rho r^3$$

$$r = \left(\frac{3m}{4\pi\rho}\right)^{1/3}$$

$$= \left(\frac{3 \cdot (1\,\text{kg})}{4\pi(11,350 \frac{\text{kg}}{\text{m}^3}}\right)^{1/3}$$

$$= 0.0276\,\text{m}$$

Force



$$|\vec{F}_{12}| = \left| - \frac{G m_1 m_2}{r_{12}^2} \right|$$

$$F_{12} = \frac{\left(6.67\times10^{-11}\,\frac{Nm^2}{kg^2}\right)(1kg)^2}{(2\cdot 0.0276m)^2}$$

$$= 2.19\times10^{-8}N$$

$$\frac{F_{12}}{m_1 g} = \frac{2.19\times10^{-8}N}{(1kg)(9.81\,m/s^2)} = 2.23\times10^{-9}N$$

(6.3) Gauss' Law for Gravity

$$\int \vec{g} \cdot \hat{n} \, dA = -4\pi G M_{enc}$$

Choose a spherical Gaussian surface

$$4\pi r^2 g = -4\pi G M_{enc}$$

$$g = \frac{-G M_{enc}}{r^2}$$

If the earth is a uniform mass with density ρ

$$M_{enc} = \frac{4}{3}\pi \rho r^3$$

$$g = -\frac{G}{r^2}\left(\frac{4}{3}\pi \rho r^3\right)$$

$$= -\left(\frac{4\pi G}{3}\rho\right) r$$

<u>EOM</u>

$$F = +mg = m\ddot{r}$$

$$\ddot{r} - g = 0$$

$$\ddot{r} + \underbrace{\left(\frac{4\pi G}{3}\rho\right)}_{\omega^2} r = 0$$

$$\omega = \sqrt{\frac{4\pi G\rho}{3}}$$

<u>What's ρ?</u>

$$\rho = \frac{M}{\frac{4}{3}\pi R_e^3}$$

$$\omega = \sqrt{\frac{4\pi G}{3} \cdot \frac{M}{\frac{4}{3}\pi R_e^3}} = \sqrt{\frac{GM}{R_e^3}}$$

but $g = \dfrac{GM}{R_e^2}$ so

$$\omega = \sqrt{\frac{g}{R_e}} = 2\pi/T$$

$$T = 2\pi\sqrt{\frac{R_e}{g}} = 1.4 \text{ hrs}$$

$$= 2\pi\sqrt{\frac{6.4\times10^6\,\text{m}}{9.81\ \text{m/s}^2}}$$

6.5 For a circular orbit,

$$F = \frac{GmM}{r^2} = ma_c = \frac{mv^2}{r}$$

$$v = \frac{2\pi r}{T}$$ where T is the period

$$\frac{GmM}{r^2} = \frac{m(2\pi r/T)^2}{r}$$

$$T^2 = \frac{4\pi^2}{MG} r^3$$

6.9

A spherical Gaussian surface of radius r enclosed

$$M_{enc} = M_{sun} + \frac{4}{3}\pi \rho r^3$$

As before Gauss' Law for Gravity for a spherical system is

$$4\pi r^2 g = -4\pi G M_{enc}$$

$$g = -\frac{G}{r^2} M_{enc}$$

$$= -\frac{G}{r^2}\left(M_{sum} + \frac{4}{3}\pi \rho r^3\right)$$

$$= -\frac{G M_{sun}}{r^2} - \frac{4}{3}\pi G \rho r$$

$$F = mg = -\frac{G m M_{sun}}{r^2} - \frac{4}{3}\pi G m \rho r$$

(6.10) $r = r_0 e^{k\theta}$

Equation of orbit

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2} f(u^{-1})$$

$$u = \frac{1}{r} = \frac{1}{r_0} e^{-k\theta}$$

$$\frac{du}{d\theta} = -\frac{k}{r_0} e^{-k\theta} = -ku$$

$$\frac{d^2u}{d\theta^2} = \frac{k^2}{r_0} e^{-k\theta} = k^2u$$

$$k^2u + u = -\frac{1}{ml^2u^2} f(u^{-1})$$

$$f(u^{-1}) = -ml^2u^3(k^2+1)$$

$$f(r) = -\frac{ml^2(k^2+1)}{r^3}$$

Angular momentum is conserved $\ell = r^2\dot{\theta}$

$$\frac{d\theta}{dt} = \dot{\theta} = \frac{\ell}{r^2} = \frac{\ell}{r_0^2} e^{-2k\theta}$$

$$\int_0^\theta d\theta\, e^{2k\theta} = \frac{\ell}{r_0^2}\int_0^t = \frac{\ell}{r_0^2} t$$

$$\frac{1}{2k}\left(e^{2k\theta} - 1\right) = \frac{\ell}{r_0^2} t$$

$$e^{2k\theta} = \frac{2k\ell}{r_0^2} t + 1$$

$$2k\theta = \ln\left(\frac{2k\ell t}{r_0^2} + 1\right)$$

$$\theta(t) = \frac{1}{2k}\ln\left(\frac{2k\ell t}{r_0^2} + 1\right)$$

6.16 $\quad \varepsilon = 0.967 \qquad r_0 = 55{,}000{,}000\ mi$

Perihelion $\quad r_0 = \dfrac{\alpha}{1+\varepsilon}$

$$\alpha = r_0(1+\varepsilon) = 108{,}190{,}000\ mi$$

$$\alpha = \frac{m\ell^2}{k} \qquad k = m M_\odot G$$

$$\ell^2 = \frac{k\alpha}{m} = M_\odot G \alpha$$

Mass of sun $\quad M_\odot = 2\times10^{30}\ kg$

$$\ell^2 = (2\times10^{30}\ kg)\left(6.67\times10^{-11}\ \frac{Nm^2}{kg^2}\right)\left(1.08\times10^{8}\ mi \cdot \frac{1609\ m}{mi}\right)$$

$$= 2.32\times10^{31}\ \frac{m^4}{s^2}$$

Latus Rectum

$$\alpha = (1-\varepsilon^2)a$$

$$r_0 = \frac{(1-\varepsilon^2)\,a}{1+\varepsilon} = \frac{\alpha}{1+\varepsilon}$$

$$= (1-\varepsilon)\,a$$

<u>Semi-major axis</u>

$$a = \frac{r_0}{1-\varepsilon} = \frac{55{,}000{,}000\ \text{mi}}{1-0.967} \cdot \frac{1609\ \text{m}}{1\ \text{mi}}$$

$$= 2.68\times10^{12}\ \text{m}$$

<u>Period</u>

$$\tau^2 = \frac{4\pi^2}{G M_\odot}\,a^3$$

$$\tau = \left(\frac{4\pi^2 \cdot (2.68\times10^{12}\ \text{m})^3}{\left(6.67\times10^{-11}\ \frac{\text{N m}^2}{\text{kg}^2}\right)\left(2\times10^{30}\ \text{kg}\right)} \right)^{1/2}$$

$$= 2.39\times10^{9}\ \text{s} \cdot \frac{1\ \text{year}}{365\cdot24\cdot60\cdot60}$$

$$\boxed{= 75.6\ \text{years}}$$

(b) At peri or apihelion $\ell = rv$

since $\vec{r} \perp \vec{v}$.

$$\ell = 4.82 \times 10^{15} \ \frac{m^2}{s}$$

At perihelion,

$$\ell = v r_0$$

$$v = \frac{\ell}{r_0} = \frac{4.82 \times 10^{15} \ \frac{m^2}{s}}{55{,}000{,}000 mi \cdot \frac{1609 m}{1 mi}}$$

$$= 5.4 \times 10^4 \ m/s$$

At apihelion

$$r_1 = (1+\varepsilon) a = 5.27 \times 10^{12} m$$

$$v_1 = \frac{\ell}{r_1} = \frac{4.82 \times 10^{15} \ \frac{m^2}{s}}{5.27 \times 10^{12} m}$$

$$= 9.1 \times 10^2 \ m/s$$

6.18 As before, at apihelion and perihelion, $l = rv$ so the product of the velocities at api and peri helion are:

$$v_0 v_1 = \frac{l^2}{r_0 r_1}$$

The period is

$$\tau^2 = \frac{4\pi^2 a^4}{l^2}(1-\varepsilon^2) \qquad (6.6.4)$$

$$r_0 = (1-\varepsilon)a \qquad r_1 = (1+\varepsilon)a$$

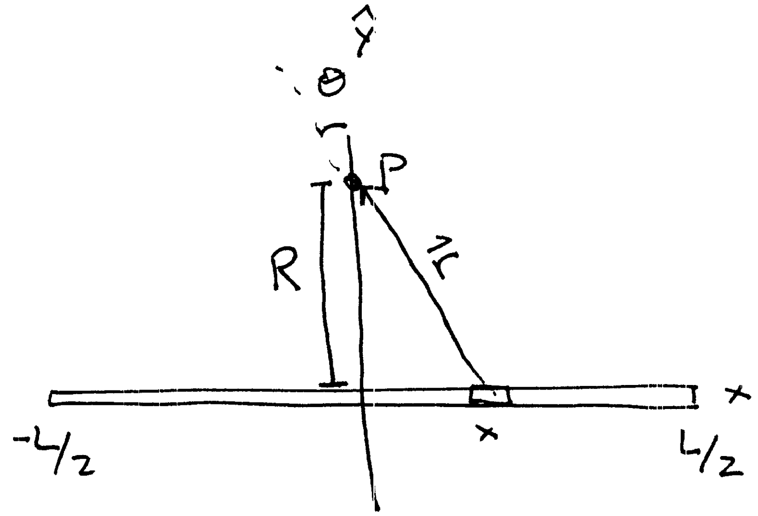
$$r_0 r_1 = (1-\varepsilon^2)a^2$$

$$\tau^2 = \frac{4\pi^2 a^2}{l^2} r_0 r_1$$

$$r_0 r_1 = \frac{l^2 \tau^2}{4\pi^2 a^2}$$

$$v_0 v_1 = \frac{l^2}{\frac{l^2\tau^2}{4\pi^2 a^2}} = \frac{4\pi^2 a^2}{\tau^2} = \left(\frac{2\pi a}{\tau}\right)^2$$

E1



The field of a small chunk of the rod dM is

$$d\vec{g} = \frac{-G\,dM}{r^2}\cos\theta\,\hat{y}$$

since by symmetry the field must point in the $-\hat{y}$ direction.

$$r = \sqrt{x^2 + R^2}$$

where R is the distance from the rod to P.

$$\cos\theta = \frac{R}{r} = \frac{R}{\sqrt{x^2+R^2}}$$

So

$$d\vec{g} = -\frac{G\,dM\,R}{\left(\sqrt{x^2+R^2}\right)^3}\,\hat{y}$$

$$dM = \lambda dx$$

$$\vec{g} = \int d\vec{g} = \hat{y}\int_{-L/2}^{L/2} \frac{-GR\lambda\, dx}{(x^2+R^2)^{3/2}}$$

$$= -\hat{y}\, GR\lambda \left(\frac{1}{R^2} \frac{x}{\sqrt{x^2+R^2}} \right)\Bigg|_{-L/2}^{L/2}$$

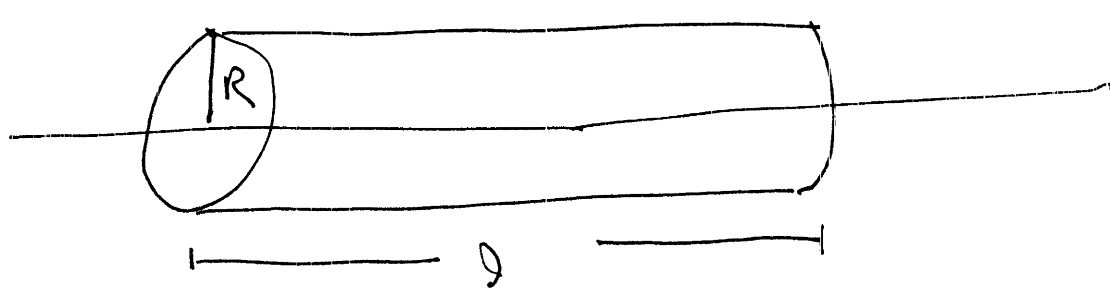
$$= -\frac{2G\lambda}{R} \frac{L/2}{((L/2)^2+R^2)^{3/2}}\, \hat{y}$$

$$\vec{g} = -6.67\times10^{-8} \frac{N}{kg}\, \hat{y}$$

As the rod becomes infinitely long $L \to \infty$

and $\quad \vec{g} \to \dfrac{-2G\lambda}{R}\, \hat{y}$

Use a cylindrical Gaussian surface co-axial with the mass



The mass enclosed by the surface is $M_{enc} = \lambda l$

Gauss' Law for Gravity

$$\int_S \overline{g} \cdot \hat{n} \, dA = -\frac{M_{enc}}{\varepsilon_g} = -4\pi G M_{enc}$$

$$= g\underbrace{(2\pi R l)}_{\text{surface area}}$$

Gaussian surface

$$g(2\pi R l) = -4\pi G(\lambda l)$$

$$g = -\frac{2G\lambda}{R}$$

The gravitational potential of the rod is given by

$$\Phi = \int d\Phi = -\int \frac{G\,dM}{r}$$

$$= -\int_{-L/2}^{L/2} \frac{G\lambda\,dx}{\sqrt{x^2+R^2}}$$

$$= +G\lambda\left(\ln\left[\frac{-L+\sqrt{L^2+4R^2}}{L+\sqrt{L^2+4R^2}}\right.\right)$$

Let $R=y$

$$\Phi(y) = +G\lambda\ln\left[\frac{-L+\sqrt{L^2+4R^2}}{L+\sqrt{L^2+4R^2}}\right]$$

$$\vec{g} = -\frac{d}{dy}\Phi = \frac{-2G\lambda L}{R\sqrt{L^2+4R^2}}\,\hat{y}$$